Note on the Application of the Pauli Ring to Form the Metric Tensor in the General Theory of Relativity

by

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1. Introduction

This paper *) will discuss the differential geometry of four dimensional spaces needed in the General Theory of Relativity. The need for it arose in connection with the attempt to introduce in such spaces a notion which would induce the metric and the affine connection in a natural way, and would make it possible to consider an invariant derivative instead of a covariant one. That is why we introduce the notion of a metric vector which has components in a certain linear space. If we consider the four dimensional Clifford Algebra called the Pauli Ring as this linear space, we obtain some interesting results which cannot be obtained in any other way. The metric tensor in this case always has a hyperbolic character suggesting a possible application to the General Theory of Relativity. The purpose of this paper is to show the fundamental dependences induced by the metric vector on the components in the Pauli Ring.

A connection between this calculus and the equations of Mathematical Physics is shown in the last part. We may add that similar ideas are well known in mathematics, as can be seen from the papers quoted.

2. The Pauli Ring

Let us consider a real linear ring $\mathfrak{P}$ generated by three elements $\sigma_1, \sigma_2, \sigma_3$ such that

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$$

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If we denote by i the product $\sigma_1\sigma_2\sigma_3$ then $i^2=-1$ and $\mathfrak{P}$ may be considered as a complex linear four dimensional ring such that

$$\sigma_1\sigma_2=i\sigma_3; \qquad \sigma_2\sigma_3=i\sigma_1; \qquad \sigma_3\sigma_1=i\sigma_2.$$

An arbitrary element $\mathbf{a}\,\epsilon\,\mathfrak{P}$ may be represented in the form

$$\mathbf{a}=a^l\sigma_l \qquad l=0,1,2,3,$$

where $\sigma_0=1$ and a^l are arbitrary complex numbers. We introduce the involutions in $\mathfrak{P}$ in the following two ways:

$$\mathbf{a}^*=a^0-a^1\sigma_1-a^2\sigma_2-a^3\sigma_3, \qquad \text{and} \qquad \overline{\mathbf{a}}=\overline{a}^l\sigma_l,$$

where $\mathbf{a}=a^l\sigma_l$ and $\overline{a}^l$ are complex adjoint with a^l.

The following identities are satisfied:

$$(\mathbf{a}^*)^*=\mathbf{a}; \quad (\overline{\overline{\mathbf{a}}})=\mathbf{a}; \quad \overline{(\mathbf{a}^*)}=(\overline{\mathbf{a}})^* \quad \text{for every} \quad a\,\epsilon\,\mathfrak{P} \quad \text{and} \quad (\mathbf{ab})^*=\mathbf{b}^*\mathbf{a}^*;$$

$$\overline{\mathbf{ab}}=\overline{\mathbf{b}}\,\overline{\mathbf{a}} \qquad \mathbf{ab}^*+\mathbf{ba}^*=\mathbf{a}^*\mathbf{b}+\mathbf{b}^*\mathbf{a}=2\,(a^0b^0-a^1b^1-a^2b^2-a^3b^3),$$

where $\mathbf{a}=a^l\sigma_l$ and $\mathbf{b}=b^l\sigma_l$.

Now we may introduce in $\mathfrak{P}$ the complex valued scalar product in the following way

$$(\mathbf{a},\mathbf{b})=\tfrac{1}{2}\,(\mathbf{ab}^*+\mathbf{ba}^*).$$

It is easy to see that

$$(\mathbf{a},\mathbf{b})=(\mathbf{b},\mathbf{a}); \qquad (\mathbf{a}^*,\mathbf{b})=(\mathbf{a},\mathbf{b}^*) \qquad \text{and} \qquad (\overline{\mathbf{a}},\overline{\mathbf{b}})=\overline{(\mathbf{a},\mathbf{b})}.$$

3. The concept of P-vector and P-tensor (definitions)

We shall call a covariant *) four dimensional vector with components $\mathbf{g}_i\,\epsilon\,\mathfrak{P}$, $i=0,1,2,3$: a *covariant P-vector* if $\det(\mathbf{g}_i,\mathbf{g}_j)\neq 0$ and $\overline{\mathbf{g}}_i=\mathbf{g}_i$.

We shall call a contravariant **) four dimensional vector with components $g^i\,\epsilon\,P$ $i=0,1,2,3$, a *contravariant P-vector* if $\det(\mathbf{g}^i,\mathbf{g}^j)\neq 0$ and $\overline{\mathbf{g}}^i=\mathbf{g}^i$.

We shall call a covariant P-vector $\mathbf{g}_i$ and a contravariant P-vector $\mathbf{g}^i$, adjoint if $(\mathbf{g}^k,\mathbf{g}_l)=\delta_l^k$.

It is evident that to every covariant or contravariant P-vector exactly one adjoint P-vector can be found, contravariant or covariant

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*) *i. e.* a vector which transforms respectively together with affine transformations of fixed four dimensional linear space.

respectively. It is easy to see that if $\mathbf{g}_i$ and $\mathbf{g}^i$ are adjoint covariant and contravariant P-vectors, then

$$\mathbf{g}^k = (\mathbf{g}^k, \mathbf{g}^j)\,\mathbf{g}_j \qquad \text{and} \qquad \mathbf{g}_l = (\mathbf{g}_l, \mathbf{g}_i)\,\mathbf{g}^i.$$

We shall call the pair composed of the covariant and contravariant adjoint P-vectors $\mathbf{g}_i$ and $\mathbf{g}^i$: a P-vector; we shall call $\mathbf{g}_i$ and $\mathbf{g}^i$ respectively the covariant and contravariant components of this P-vector.

We introduce tensors with covariant and contravariant components induced by a P-vector as follows:

$$\mathbf{g}_{i_1 i_2 \ldots i_n} \overset{\mathrm{df}}{=} \frac{1}{n!} \underbrace{\mathbf{g}_{i_1} \mathbf{g}^*_{i_2} \cdots}_{n \text{ times}} \; ; \qquad \mathbf{g}^{i_1 i_2 \ldots i_n} \overset{\mathrm{df}}{=} \frac{1}{n!} \underbrace{\mathbf{g}^{i_1} \mathbf{g}^{*i_2} \cdots}_{n \text{ times}}$$

$n = 1, 2, 3, 4$. For example:

$$\mathbf{g}_i = \mathbf{g}_i; \qquad \mathbf{g}_{ij} = \frac{1}{2}\mathbf{g}_i \mathbf{g}^*_j; \qquad \mathbf{g}_{ijk} = \frac{1}{6}\mathbf{g}_i \mathbf{g}^*_j \mathbf{g}_k; \qquad \mathbf{g}_{ijkl} = \frac{1}{24}\mathbf{g}_i \mathbf{g}^*_j \mathbf{g}_k \mathbf{g}^*_l.$$

These tensors we shall call P-*tensors*.

Every P-vector induces a symmetric tensor which we shall call the *fundamental tensor*. Its covariant and contravariant components are induced as follows:

$$g_{ij} = (\mathbf{g}_i, \mathbf{g}_j) \qquad \text{and} \qquad g^{ij} = (\mathbf{g}^i, \mathbf{g}^j),$$

where $\mathbf{g}_i$ and $\mathbf{g}^i$ are respectively covariant and contravariant components of the P-vector under consideration.

THEOREM 1. *If we put* $\check{\mathbf{g}} \overset{\mathrm{df}}{=} \varepsilon^{ijkl}\mathbf{g}_{ijkl}$ *and* $\hat{\mathbf{g}} \overset{\mathrm{df}}{=} \varepsilon_{ijkl}\mathbf{g}^{ijkl}$, *where* ε_{ijkl} *and* ε^{ijkl} *are skew symmetric and* $\varepsilon_{0123} = \varepsilon^{0123} = 1$, *then the following identities are satisfied:*

$$\check{\mathbf{g}}^* - \check{\mathbf{g}} = \hat{\mathbf{g}}^* - \hat{\mathbf{g}} = \check{\mathbf{g}} + \bar{\check{\mathbf{g}}} = \hat{\mathbf{g}} + \bar{\hat{\mathbf{g}}} = 0; \qquad \check{\mathbf{g}}\hat{\mathbf{g}} = 1,$$

$$\check{\mathbf{g}}^2 = \det \|g_{ij}\| < 0; \qquad \hat{\mathbf{g}}^2 = \det \|g^{ij}\| < 0.$$

Therefore in the case under consideration the metric tensor has a hyperbolic character.

THEOREM 2. *Let us consider an arbitrary P-vector with covariant and contravariant components equal to* $\mathbf{g}_i$ *and* $\mathbf{g}^i$ *respectively. If* $g_{00} > 0$, *then the three-dimensional tensor* $h^{\alpha\beta} = -g^{\alpha\beta}$; $\alpha, \beta = 1, 2, 3$ *is positively defined. Hence if we put*

$$h_{\alpha\beta} \overset{\mathrm{df}}{=} \frac{g_{0\alpha}\, g_{0\beta}}{g_{00}} - g_{\alpha\beta},$$

then

$$h_{\alpha\gamma} h^{\gamma\beta} = \delta^\beta_\alpha \qquad for \qquad \alpha, \beta, \gamma = 1, 2, 3.$$

4. A correspondence between P-scalars and real skew symmetric tensors

Let us denote by $\overset{(n)}{\mathfrak{B}}$, $n=1,2,3,4$, the following subsets of $\mathbf{P}$:

$$\overset{(1)}{\mathfrak{B}} = \underset{x \in \mathfrak{P}}{E} (\mathbf{x} - \bar{\mathbf{x}} = 0); \qquad \overset{(2)}{\mathfrak{B}} = \underset{x \in \mathfrak{P}}{E} (\mathbf{x} + \mathbf{x}^* = 0);$$

$$\overset{(3)}{\mathfrak{B}} = \underset{x \in \mathfrak{P}}{E} (\mathbf{x} + \bar{\mathbf{x}} = 0); \qquad \overset{(4)}{\mathfrak{B}} = \underset{x \in \mathfrak{P}}{E} (\mathbf{x} + \bar{\mathbf{x}} = 0, \ \mathbf{x} = \mathbf{x}^*).$$

Let $\overset{(n)}{\mathfrak{U}}$, $n=1,2,3,4$, denote the set of all skew symmetric tensors X of n-th rank with real covariant or contravariant components $X_{i,i_2\ldots i_n}$ or $X^{i,i_2\ldots i_n}$ respectively, such that

$$X_{i,i_2\ldots i_n} = g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_n j_n} X^{j_1 i_2 \ldots j_n},$$

or in the equivalent way

$$X^{i_1 i_2 \ldots i_n} = g^{i_1 j_1} g^{i_2 j_2} \cdots g^{i_n j_n} X_{j_1 j_2 \ldots j_n}.$$

THEOREM 3. *There is a one-to-one correspondence between skew symmetric tensors with real components and P-scalars. This correspondence can be realised by the transformation:*

$$\overset{(n)}{f}(X) = X_{i_1 i_2 \ldots i_n} \mathbf{g}^{i_1 i_2 \ldots i_n} = X^{i_1 i_2 \ldots i_n} \mathbf{g}_{i_1 i_2 \ldots i_n}$$

which maps one-to-one $\overset{(n)}{\mathfrak{U}}$ onto $\overset{(n)}{\mathfrak{B}}$, $n=1,2,3,4$.

If we denote by $\overset{(n)}{\varphi}$ the inverse mapping of $\overset{(n)}{f}$, then the following identities are satisfied:

$$\overset{(n)}{\varphi}_{i_1 i_2 \ldots i_n}(\mathbf{x}) = \frac{n!}{2} \left[(\mathbf{x}, \mathbf{g}_{i_1 i_2 \ldots i_n}) + \overline{(\mathbf{x}, \mathbf{g}_{i_1 i_2 \ldots i_n})} \right]$$

and

$$\overset{(n)}{\varphi}{}^{i_1 i_2 \ldots i_n}(\mathbf{x}) = \frac{n!}{2} \left[(\mathbf{x}, \mathbf{g}^{i_1 i_2 \ldots i_n}) + \overline{(\mathbf{x}, \mathbf{g}^{i_1 i_2 \ldots i_n})} \right],$$

where

$$\overset{(n)}{\varphi}_{i_1 i_2 \ldots i_n}(\mathbf{x}) \ and \ \overset{(n)}{\varphi}{}^{i_1 i_2 \ldots i_n}(\mathbf{x}),$$

covariant and contravariant components of $\overset{(n)}{\varphi}(\mathbf{x}) \in \overset{(n)}{\mathfrak{U}}$.

5. Covariant and invariant derivation

Let Ω be a fixed four dimensional domain. We shall denote its point by x^i, $i=0,1,2,3$.

Let us consider a P-vector field defined in this domain, i. e. the P-vector valued function defined in the whole of Ω. The covariant and

contravariant components $\mathbf{g}_i$ and $\mathbf{g}^i$ of this P-vector field are respectively covariant or contravariant P-vector valued functions with adjoint values, i. e. $(\mathbf{g}_i, \mathbf{g}^j) = \delta_i^j$.

Every P-vector field usually induces P-tensor fields $\mathbf{g}_{i_1 i_2 \dots i_n}, \mathbf{g}^{i_1 i_2 \dots i_n}$ *) and the metric tensors g_{ij}, g^{ij} as a field of the fundamental tensor.

We now introduce the covariant derivate V_k in such a way that the derivatives of the P-tensors and of the metric tensors equal zero. This can be done if we define the coefficients of the affine connection as follows:

$$\Gamma_{kl}^i \overset{\text{df}}{=} (\partial_k \mathbf{g}_l, \mathbf{g}^i) \left(\frac{\partial \mathbf{g}_l}{\partial \mathbf{x}^k}, g^i \right).$$

This implies that

$$V_k \mathbf{g}_{i_1 i_2 \dots i_n} = 0; \qquad V_k \mathbf{g}^{i_1 i_2 \dots i_n} = 0; \qquad \sqrt{R_{klm}^i = 0,}$$
$$V_k g_{ij} = 0; \qquad V_k g^{ij} = 0.$$

where R_{klm}^i is the Riemann curvature tensor. If we denote by S_{kl}^i the *tensor of torsion* then:

$$S_{kl}^i = \tfrac{1}{2}(\Gamma_{kl}^i - \Gamma_{lk}^i) = (\mathbf{g}^i, \mathbf{s}_{kl}),$$

where

$$\mathbf{s}_{kl} = \mathbf{g}_i S_{kl}^i = \tfrac{1}{2}(\partial_k \mathbf{g}_l - \partial_l \mathbf{g}_k);$$

we shall name the P-*tensor of torsion*.

We can also prove the following

REMARK 1. *The condition* $S_{kl}^i = 0$ *is equivalent to the existence of a* P-*scalar* $\mathbf{g}$, *such that* $\mathbf{g}_i = \partial_i \mathbf{g}$.

REMARK 2. *If we denote* $\Gamma_{(kl)}^i = \tfrac{1}{2}(\Gamma_{kl}^i + \Gamma_{lk}^i)$, *then*

$$\left\{ \begin{matrix} s \\ m k \end{matrix} \right\} - \Gamma_{(mk)}^s = g_{rk} g^{ls} S_{ml}^r + g_{rm} g^{ls} S_{kl}^r,$$

where $\left\{ \begin{matrix} s \\ m k \end{matrix} \right\}$ *are Christofel symbols*.

We shall now denote by [ ] the operation of alternation. For example if $X \in \overset{2}{\mathfrak{U}}$ then

$$V_{[i} X_{jk]} = \tfrac{1}{3}(V_i X_{jk} - V_j X_{ik} - V_k X_{ji}).$$

We introduce the operation of taking the invariant derivative as follows:

$$D = \mathbf{g}^i \partial_i \quad \text{and} \quad D^* = \mathbf{g}^{*i} \partial_i.$$

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*) i. e. with a differentiable one-to-one transformation f^i of domain Ω every point of this field transforms with the affine transformation $a_j^i = \partial_j f^i$ of contact space into this point.

It can be seen that

$$DD^* = D^*D = \square = g^{ij}\frac{\partial^2}{\partial_x^i \partial_x^j}.$$

We shall assume that in the class of P-valued functions which we consider, the equation $\square\,x = a$ has a solution. The following identity is satisfied:

(1) $$D^*x = \overline{(D\overline{x^*})^*},$$

where x is a P-valued function.

THEOREM 4. *Let us consider a P-scalar $x \in \overset{(n)}{\mathfrak{B}}$. By Theorem 3 there is a $X \in \overset{(n)}{\mathfrak{U}}$ such that*

$$x = \overline{g}^{i_1 i_2 \dots i_n} X_{i_1 i_2 \dots i_n} = \overline{g}_{i_1 i_2 \dots i_n} X^{i_1 i_2 \dots i_n}.$$

Then the following identity is satisfied *):

(2) $$D x^* = g^{i_0 i_1 \dots i_n} \nabla_{[i_0} X_{i_1 i_2 \dots i_n]} + g^{i_2 i_3 \dots i_n} \nabla^i X_{i_1 i_2 \dots i_n}$$

$$= g_{i_0 i_1 \dots i_n} \nabla^{[i_0} X^{i_1 i_2 \dots i_n]} + g_{i_2 i_3 \dots i_n} \nabla_{i_1} X^{i_1 i_2 \dots i_n}.$$

REMARK 3. *From identity (2) an alternated covariant derivative and contracted covariant derivative can be found as follows:*

$$\nabla_{[i_0} X_{i_1 i_2 \dots i_n]} = \varphi_{i_0 i_1 \dots i_n}\!\left(\overset{(n)}{V}(x)\right),$$

$$\nabla^{[i_0} X^{i_1 i_2 \dots i_n]} = \varphi^{i_0 i_1 \dots i_n}\!\left(\overset{(n)}{V}(x)\right),$$

$$\nabla^{i_1} X_{i_1 i_2 \dots i_n} = \varphi_{i_2 \dots i_n}\!\left(\overset{(n)}{W}(x)\right),$$

$$\nabla_{i_1} X^{i_1 i_2 \dots i_n} = \varphi^{i_2 \dots i_n}\!\left(\overset{(n)}{W}(x)\right),$$

where $x = f(X)$ and

$$\overset{(1)}{V}(x) = \tfrac{1}{2}\left(Dx^* - (Dx^*)^*\right), \qquad \overset{(1)}{W}(x) = \tfrac{1}{2}\left(Dx^* + (Dx^*)^*\right),$$

$$\overset{(2)}{V}(x) = \tfrac{1}{2}\left(Dx^* - \overline{Dx^*}\right), \qquad \overset{(2)}{W}(x) = \tfrac{1}{2}\left(Dx^* + \overline{Dx^*}\right),$$

$$\overset{(3)}{V}(x) = \tfrac{1}{2}\left(Dx^* - (Dx^*)^*\right), \qquad \overset{(3)}{W}(x) = \tfrac{1}{2}\left(Dx^* + (Dx^*)^*\right),$$

$$\overset{(4)}{V}(x) = 0, \qquad\qquad\qquad \overset{(4)}{W}(x) = Dx^*.$$

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*) For $n = 1$: $Dx^* = g^{i_0 i_1}\nabla_{[i_0} X_{i_1]}$ and $\nabla^{i_1}X_{i_1} = \tfrac{1}{2}\left(Dx^* + (Dx^*)^*\right)$.

6. A correspondence between tensor and P-scalar differential equations

Let us consider a system of differential equations similar to Maxwell's equations:

$$(3) \qquad \begin{cases} V_{[i}F_{jk]}=0, \\ V^i\,F_{ij}\ =I_j, \end{cases} \quad \text{where} \quad F_{ij}+F_{ji}=0$$

we have from (2):

$$\boldsymbol{D}f^* = \mathbf{g}^{ijk}\,V_{[i}F_{jk]} + \mathbf{g}^k\,V^i F_{ik} = \mathbf{g}^j\,I_j = \mathbf{i}, \quad \text{where} \quad \mathbf{f}=\overline{\mathbf{g}}^{ij}F_{ij}=\overline{f(F)}.$$

Therefore the invariant equation equivalent to (3) is

$$(4) \qquad\qquad \boldsymbol{D}\mathbf{f}^*=\mathbf{i}, \quad \text{where} \quad \mathbf{f}+\mathbf{f}^*=0 \quad \text{and} \quad \mathbf{i}=\overline{\mathbf{i}}.$$

Evidently there is an $\mathbf{a}=\overline{\mathbf{a}}$ such that $\mathbf{f}^*=\boldsymbol{D}^*\mathbf{a}$ and we have $\boldsymbol{D}^*\mathbf{a}+ + (\boldsymbol{D}^*\mathbf{a})^*=0$ which implies $V^i A_i=0$ where $A_i=(\mathbf{a},\mathbf{g}_i)$.

Let us consider now a system of differential equations similar to Proca's equations:

$$(5) \qquad \begin{cases} V_{[i}F_{jk]}=0, \\ V^i\,F_{ij}\ =\mu A_j, \\ V_{[i}A_{j]}=\mu F_{ij}, \end{cases} \quad \text{where} \quad F_{ij}+F_{ji}=0.$$

We have from (1) and (2)

$$\boldsymbol{D}\mathbf{f}^* = \mathbf{g}^{ijk}\,V_{[i}F_{jk]} + \mathbf{g}^k\,V^i F_{ij} = \mu \mathbf{g}^k A_k = \mu \mathbf{a}$$

$$\boldsymbol{D}^*\mathbf{a} = (\overline{\mathbf{g}}^{ij}V_{[i}A_{j]}) + V^i A_i = \mu \mathbf{f}^*, \quad \text{where} \quad \mathbf{f}=\overline{\mathbf{g}}^{ij}F_{ij}, \quad \mathbf{a}=\mathbf{g}^k A_k.$$

Therefore the equations equivalent to (5) are

$$(6) \qquad\qquad \left.\begin{array}{l} \boldsymbol{D}\mathbf{f}^*=\mu\mathbf{a} \\ \boldsymbol{D}^*\mathbf{a}=\mu\mathbf{f} \end{array}\right\}, \quad \text{where} \quad \mathbf{f}+\mathbf{f}^*=0 \quad \text{and} \quad a=a^*.$$

These equations are equivalent to Dirac's equations. In fact, let us consider the equations:

$$(7) \qquad\qquad \begin{array}{l} (\boldsymbol{D}+\mathbf{a})\mathbf{x}\ =\mu\mathbf{y} \\ (\boldsymbol{D}+\mathbf{a})^*\mathbf{y}=\mu\mathbf{x}. \end{array}$$

It $\mathbf{g}_i=\boldsymbol{\sigma}_i$ and if we represent (7) in matrices in such a way that

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \qquad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

and

$$\mathbf{x}-\mathbf{y}=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \qquad \mathbf{x}+\mathbf{y}=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

then we obtain an equation equivalent to (7), analogous to Dirac's

equation, with an unknown wave function $\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \psi_1 \\ \psi_2 \end{pmatrix}$, where $\mathbf{a} = A_i \mathbf{g}^i$ is the

vector potential of the electromagnetic field.

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MATHÉMATIQUE

Réduction des axiomes de congruence de Hilbert

par

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Les axiomes de congruence énoncés par Hilbert ([1], Chap. I, § 5) ne sont pas indépendants. L'axiome III_3 ainsi qu'une partie de l'axiome III_4 sont superflus.

Pour les différents axiomes de congruence ou leurs parties, nous introduirons les notations suivantes: III_1, III_2, III_5, tels qu'ils ont été énoncés par Hilbert, et, en plus:

III_4^* — Étant donnés un angle $\sphericalangle ABC$ et un demi-plan limité par la droite KL, $K \neq L$, passant par un point M qui n'est pas sur KL, il existe au moins un point N, appartenant à ce demi-plan, tel que $\sphericalangle ABC \cong \sphericalangle KLN$.

III_4^{**} — Étant donnés un angle $\sphericalangle ABC$ et un demi-plan limité par la droite KL, $K \neq L$, passant par un point M qui n'est pas sur KL, il existe au plus un point N, appartenant à ce demi-plan, tel que $\sphericalangle ABC \cong \sphericalangle KLN$.

Nous rejetons l'axiome III_3 de Hilbert relatif à la congruence des sommes de segments, ainsi que la thèse contenue dans l'axiome III_4:

III_4^{***} — Tout angle est congruent à lui-même.

Chacun des axiomes III_1, III_2, III_4^*, III_4^{**} et III_5 est indépendant des autres, il est aussi indépendant des axiomes d'incidence et d'ordre. Pour le démontrer nous construisons des modèles d'après le schéma dans la table à la page suivante.

Nous appelons points, droites et plans des points, droites et plans de l'espace cartésien à trois dimensions; par contre, les relations de congruence $AB \cong A'B'$ et $\sphericalangle ABC \cong \sphericalangle A'B'C'$ sont interprétées comme le montre le tableau.

Dans le cas des axiomes III_1, III_2, III_4^* et III_4^{**} ces modèles prouvent aussi que celles-ci sont indépendantes des thèses III_3 et III_4^{***}. Dans le cas III_5 le modèle donné dans la table ne suffit pas pour établir

TABLE

Indépendance de la thèse	Congruence des segments	Congruence des angles
III_1	identité des segments	congruence dans le sens usuel
III_2	mesure $AB = 2$ mesure $A'B'$	„
III_4^*	congruence dans le sens usuel	identité des angles
III_4^{**}	„	deux angles arbitraires sont congruents
III_5	„	mesure $\sphericalangle ABC = 2$ mesure $\sphericalangle A'B'C'$

l'indépendance, mais il existe un modèle convenable donné par Hilbert ([1], Chap. II, § 11) *).

La preuve que les thèses rejetées III_3 et III_4^{***} sont des conséquences des axiomes d'incidence, d'ordre ainsi que des axiomes III_1, III_2, III_4^*, III_4^{**} et III_5, est beaucoup plus compliquée.

Nous n'exposerons ici que les idées principales, en ne développant plus en détail que quelques fragments **).

On démontre sans aucune modification que la congruence des segments est une relation réflexive, symétrique et transitive, que les segments peuvent être transportés univoquement, ainsi que les deux premiers cas de congruence des triangles ([1], Chap. I, § 5 et 6, Théorèmes 12 et 13).

Pour démontrer la thèse III_4^{***}, nous supposerons donné l'angle $\sphericalangle BAC$. En appliquant III_4^*, III_1 et la première proposition sur la congruence des triangles, nous construisons le triangle $\triangle BAC'$ symétrique (et congruent) par rapport au triangle $\sphericalangle BAC$. La droite CC' coupe la droite AB en un point D. Dans le cas où D est situé sur le côté AB de l'angle donné, nous avons

$$\triangle ABC \cong \triangle ABC',$$
$$\sphericalangle C \cong \sphericalangle C',$$
$$\triangle ACC' \cong \triangle AC'C,$$
$$\sphericalangle CAC' \cong \sphericalangle C'AC = \sphericalangle CAC',$$
$$\triangle CAC' \cong \triangle CAC',$$
$$\sphericalangle C \cong \sphericalangle C,$$
$$\triangle ACB \cong \triangle ACB,$$
$$\sphericalangle BAC \cong \sphericalangle BAC.$$

*) Un modèle beaucoup plus simple a été donne par R. Walker ([2], p. 169) qui a aussi démontré l'indépendance des axiomes III_1, III_2, III_4^* et III_4^{**} (p. 166—169).

**) Le texte complet du travail, contenant aussi une étude de l'indépendance des axiomes d'incidence et d'ordre, paraîtra dans les Annales Universitatis M. Curie-Skłodowska, Sectio A, vol. IX.

Le cas où le point D se confond avec le point A, ou est situé sur le prolongement du côté AB, ne présente pas de nouvelles difficultés.

Il nous est possible maintenant de donner des preuves immédiates des propositions suivantes:

Dans un triangle isoscèle les angles adjacents à la base sont mutuellement congruents.

Si dans un triangle $\triangle ABC$ les côtés AB et BC sont congruents et si le point C' est situé entre B et C, les segments AC' et BC' ne sont pas congruents (III_4^{**}).

En définissant un angle droit comme celui qui est congruent à l'un de ses angles adjacents nous démontrons qu'un angle droit et chacun de ses angles adjacents sont mutuellement congruents. Soit $\sphericalangle AOB$ un angle congruent à son angle adjacent $\sphericalangle AOB'$ et soit $OA \simeq OB \simeq OB'$. On prouve d'abord que $AB \simeq AB'$, $\sphericalangle BB'A \simeq \sphericalangle B'BA$, $\triangle OB'A \simeq \triangle OBA$ et $\sphericalangle AOB' \simeq \sphericalangle AOB$. Ensuite on construit un triangle $\triangle BB'A'$ congruent au triangle $\triangle BB'A$ du côté opposé de la droite BB' et, moyennant la proposition précédente, on prouve que la droite AO passe par le point A'. Évidemment $\triangle ABO \simeq \triangle A'BO$; donc $\sphericalangle AOB = \sphericalangle A'OB$.

Dans la suite, le théorème suivant va jouer un rôle essentiel: Dans un triangle rectangle $\triangle ABC$ le côté AC de l'angle droit ne peut être congruent à l'hypoténuse. En effet, si $\triangle ABD$ est symétrique par rapport au triangle $\triangle ABC$ et si E désigne un point de la demi-droite CB tel que $CD \simeq CE$, on constate aisément que $AC \simeq BC$ entraîne $AD \simeq AE$, ce qui est impossible, car $CD \simeq CE$.

Comme conséquence on obtient la proposition: la hauteur correspondant à la base d'un triangle isoscèle la divise en deux parties égales; moyennant cette proposition on prouve sans difficulté la proposition du deltoïde ([1], Chap. I, § 6, Théorème 17), qui entraîne comme conséquence la proposition relative à la congruence des triangles ayant leurs côtés congruents chacun à chacun, et enfin on établit que la congruence des angles est symétrique et transitive ([1], Chap. I, § 6, Théorèmes 18 et 19).

En profitant de ces propositions nous prouvons que sur un segment quelconque AB pris pour base on peut construire un triangle isoscèle (il sera déterminé par deux triangles congruents $\triangle ABC$ et $\triangle BAC'$ contenus dans un demi-plan). Il résulte de là que tout segment a un milieu. C'est tout ce dont nous avons besoin pour établir la thèse III_3:

Si les segments contigus A_1B_1 et B_1C_1 sont congruents aux segments contigus A_2B_2 et B_2C_2 respectivement, on a $A_1C_1 \simeq A_2C_2$.

La démonstration se fait en plusieurs étapes: Supposons d'abord que $A_1 = A_2$ et que C_2 ne soit pas sur la droite A_1B_1. Considérons la hauteur A_1D du triangle isoscèle $\triangle B_1B_2A_1$ et son intersection E avec la droite B_2C_1. Évidemment $\triangle A_1B_1D \simeq \triangle A_1B_2D$, $\triangle A_1B_1E \simeq \triangle A_1B_2E$ et $\triangle DB_1E = DB_2E$. Sur B_1E nous transportons le segment $B_1F \simeq B_2C_1$.

Puisque $\triangle A_1 B_2 C_1 = \triangle A_1 B_1 F$, le point F doit être sur $A_1 C_2$ et $A_1 C_1 \cong A_1 F$. De la congruence des triangles $\triangle B_1 B_2 C_1 \cong \triangle B_2 B_1 F$ il résulte que $B_1 C_1 \cong$ $\cong B_2 F$ et $C_2 = F$, c'est-à-dire $A_1 C_1 \cong A_1 C_2$.

En prenant les segments auxiliaires $A_1 B_3$, $B_3 C_3$ sur une droite qui ne passe ni par B_1, ni par B_2, nous pouvons nous débarasser de la condition que B_2 n'est pas situé sur la droite $A_1 B_1$.

Nous supposons maintenant que les points considérés sont situés sur une droite dans l'ordre suivant: C_1, B_1, A_1, A_2, B_2, C_2. Si $A_1 \neq A_2$, alors, en désignant par O le milieu du segment $A_1 A_2$, nous aurons

$$OA_1 \cong OA_2,$$

$$A_1 B_1 \cong A_2 B_2,$$

et de là

$$OB_1 \cong OB_2,$$

$$OC_1 \cong OC_2;$$

de même, on prouve aisément par l'absurde que

$$A_1 C_1 \cong A_2 C_2.$$

Il est évident que la preuve complète de la thèse III_3 se ramène aux cas que nous venons d'étudier.

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Remark on the Tensor Product of Modules

by

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Presented by K. BORSUK on April 12, 1956

A right module (over a ring $\varLambda$ with unit element, 1) is an Abelian group A, which admits right operators from $\varLambda$; thus

(i)   if $a \in A$, $\lambda \in \varLambda$, an element $a\lambda \in A$ is defined;

(ii) $a(\lambda + \lambda') = a\lambda + a\lambda'$;

(iii) $a(\lambda\lambda') = (a\lambda)\lambda'$;

(iv) $a.1 = a$.

A left module is defined similarly. Let A be a right $\varLambda$-module and B a left $\varLambda$-module. Then the tensor product $A \otimes_\varLambda B$ (which we will abbreviate to $A \otimes B$) is the Abelian group generated by the symbols $a \otimes b$, $a \in A$, $b \in B$, subject to the relations

(i) $(a + a') \otimes b = a \otimes b + a' \otimes b$;

(ii) $a \otimes (b + b') = a \otimes b + a \otimes b'$;

(iii) $a\lambda \otimes b = a \otimes \lambda b$, $\lambda \in \varLambda$.

Notice that $\varLambda \otimes B \cong B$ under $\lambda \otimes b \to \lambda b$; here $\varLambda$ is regarded as a right $\varLambda$-module in the obvious way. Similarly $A \otimes \varLambda \cong A$. $A \otimes_\varLambda B$ may be given the structure of a module over the centre of $\varLambda$, but we will regard it merely as an Abelian group *).

A module-map $\varPhi: A \to B$ is a homomorphism which commutes with the operators. A (right) projective module is a module A such that, for any maps $\varPhi: A \to B$, $\psi: C \to B$, where ψ is onto B, there exists $\theta: A \to C$ with $\varPhi = \psi\theta$. A free module is a module A possessing a set of elements a_α such that every element of A is uniquely expressible as $\sum_\alpha a_\alpha \lambda_\alpha$, $\lambda_\alpha \in \varLambda$, where the sum is finite. Every module may evidently be represented as

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*) Notice that the definition given on p. 140 of [1] is valid only if $\varLambda$ is commutative. In the corollary at the end of this note we are concerned with this case.

the homomorphic image of a free module. From this it follows immediately that a module is projective if and only if it is a direct factor of a free module. From this remark Yoneda deduced, in [3],

THEOREM 1. *Let C be a left module and B a submodule. Then the injection map $A \otimes B \to A \otimes C$ is $(1-1)$ if A is projective.*

For the result is clearly true for free modules and, if A is a direct factor in F, then $A \otimes B$ is a direct factor in $F \otimes B$.

We will say that A has property (P) if the injection $A \otimes B \to A \otimes C$ is always $(1-1)$. Yoneda stated in [3] that the converse of Theorem 1 is open. We show here that it is false.

We will say that A is locally projective if all its finitely generated submodules are projective; we prove

THEOREM 2. *Locally projective modules have property (P).*

The proof rests on the following well-known fact on the local character of tensor products.

LEMMA 1. *Let $a_1, \ldots, a_k \in A$, $b_1, \ldots, b_k \in B$ and let $\sum_{i=1}^{k} a_i \otimes b_i = 0$ in $A \otimes B$. Then there exist finitely-generated submodules $A_0 \subseteq A$, $B_0 \subseteq B$, such that $a_i \in A_0$, $b_i \in B_0$, $i = 1, \ldots, k$, and $\sum_{i=1}^{k} a_i \otimes b_i = 0$ in $A_0 \otimes B_0$.*

For if $\sum_{i=1}^{k} a_i \otimes b_i = 0$, then $\sum_{i=1}^{k} (a_i, b_i)$ is a (finite) sum of expressions of the form $(a + a', b) - (a, b) - (a', b)$, $(a, b + b') - (a, b) - (a, b')$, $(a\lambda, b) - (a, \lambda b)$. Let A_0 be the submodule generated by the first members of all pairs (a, b) occurring in this sum, and let B_0 be the submodule generated by the second members. Then $a_i \in A_0$, $b_i \in B_0$ and $\sum_{i=1}^{k} a_i \otimes b_i = 0$ in $A_0 \otimes B_0$.

We now prove Theorem 2. Let C be a left module, $B \subseteq C$. Let $\sum_{i=1}^{k} a_i \otimes b_i \in A \otimes B$ and let $\sum_{i=1}^{k} a_i \otimes b_i = 0$ in $A \otimes C$. Then there exists a finitely-generated submodule A_0 such that $\sum_{i=1}^{k} a_i \otimes b_i \in A_0 \otimes B$ and $\sum_{i=1}^{k} a_i \otimes b_i = 0$ in $A_0 \otimes C$. If A is locally projective, A_0 is projective. Thus $\sum_{i=1}^{k} a_i \otimes b_i = 0$ in $A_0 \otimes B$ and so, a fortiori, in $A \otimes B$.

Now let $\Lambda = Z$, the ring of integers. Then every projective module is free, and locally projective modules are locally free. Since not every locally free Abelian group is free (for example, the group of rationals), it follows that a module may have property (P) without being projective.

On the other hand, not every projective module is locally projective. Let Λ be the ring of polynomials in two variables x, y over an arbitrary field. Then Λ is free, and thus projective, as a module. Consider the

submodule **) A generated by x, y. These generators are subject to the relation $yx - xy = 0$. Moreover, every relation between them is a consequence of this one. Let S be a free Λ-module on two generators s_1, s_2, and let T be an isomorphic copy of A with generators t_1, t_2, subject to $yt_1 - xt_2 = 0$. Let $\varphi: A \to T$ be given by $\varphi(x) = t_1$, $\varphi(y) = t_2$ and let $\psi: S \to T$ be given by $\psi(s_1) = t_1$, $\psi(s_2) = t_2$. If we wish to define $\theta: A \to S$ such that $\psi\theta = \varphi$, we must map x to $s_1 + f(ys_1 - xs_2)$ and y to $s_2 + g(ys_1 - xs_2)$, where $f, g \in \Lambda$. But then $yx - xy$ is mapped to $ys_1 + yf(ys_1 - xs_2) - xs_2 - xg(ys_1 - xs_2)$. This can only be zero if $y + y^2f - xyg = 0$, $x - x^2g + xyf = 0$, that is, if $xg - yf = 1$, which is impossible. Thus A is not projective, and it is untrue in general that property (P) characterises locally projective modules. We know, however, that if Λ is a ring without zero divisors in which every (right) ideal is principal, then a submodule of a (right) free module is free, so that projective modules are free and free modules are locally free. Thus it remains possible that for such rings Λ, property (P) may characterise locally free modules. We prove a weaker statement

THEOREM 3. *If Λ is a ring without zero divisors in which every right and left ideal is principal, then the module A has property (P) if, and only if, it is locally free.*

This theorem rests on the

LEMMA 2. *Under the same hypotheses on Λ, A is locally free if, and only if, $a\lambda = 0$ implies $a = 0$ or $\lambda = 0$.*

The condition is obviously necessary. Now let A_0 be a finitely-generated submodule of A. It is shown in [2] that the finite basis theorem holds for Λ-modules; namely, A_0 admits a finite set of generators $a_1, \ldots, a_n$, with defining relations $a_m\lambda_m = a_{m+1}\lambda_{m+1} = \ldots = a_n\lambda_n = 0$, $1 \leqslant m \leqslant n+1$. If $a\lambda = 0$ implies $a = 0$ or $\lambda = 0$, the relations are void and A_0 is free.

We now prove the theorem. If A is not locally free, there exist $\lambda \neq 0$, $a \neq 0$ with $a\lambda = 0$. Let $C = \Lambda$, regarded as a left Λ-module, and let B be the submodule $\Lambda\lambda$. Then $A \otimes B \cong A$ under $a \otimes \lambda'\lambda \to a\lambda'$. Thus $a \otimes \lambda \neq 0$ in $A \otimes B$. On the other hand, $a \otimes \lambda = a\lambda \otimes 1 = 0$ in $A \otimes C$, so that $A \otimes B \to A \otimes C$ is not (1—1).

We close with a remark arising out of Lemma 1, which is probably well-known.

COROLLARY 1. *If A, B are modules over a principal ideal domain Λ and if a, b are free elements in A, B, then $a \otimes b$ is free in $A \otimes B$.*

Since we may factor out the torsion submodules of A, B, it is sufficient to assume that $\lambda x = 0$, $\lambda \in \Lambda$, $x \in A$ or B, implies $\lambda = 0$ or $x = 0$.

We prove first that $a \otimes b \neq 0$ in $A \otimes B$. For if $a \otimes b = 0$ in $A \otimes B$, there would exist finitely generated submodules $A_0 \subseteq A$, $B_0 \subseteq B$, such

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*) Since Λ is commutative no distinction need be drawn between right and left modules.

that $a \in A_0$, $b \in B_0$ and $a \otimes b = 0$ in $A_0 \otimes B_0$. Then canonical bases $a_0, a_1, \ldots$ for $A_0, b_0, b_1, \ldots$, for B_0 may be chosen for A_0, B_0 such that $a = \lambda a_0$, $b = \mu b_0$, whence $a \otimes b = \lambda \mu (a_0 \otimes b_0)$. But $a_0 \otimes b_0$ certainly generates a free factor in $A_0 \otimes B_0$, so that the assertion $a \otimes b = 0$ is false in $A_0 \otimes B_0$ and hence in $A \otimes B$. Since we may apply the same argument to $\lambda a \otimes b = \lambda (a \otimes b)$, for any $\lambda \in \Lambda$, it follows that $\lambda (a \otimes b) \neq 0$ if $\lambda \neq 0$, so that $a \otimes b$ is free in $A \otimes B$. Note that, since Λ is commutative, we regard $A \otimes B$ as a Λ-module, in the obvious way.

We may show in this way that the statement E. 6 on p. 159 of [1] is false. For let $\Lambda = Z$, let A be the group of rationals and let B be the direct product, for all $n \geqslant 2$, of cyclic groups, Z_n, of order n, generated by e_n. Let $e = (e_n)$. Then e is of infinite order in B. Let a be any non-zero rational. Then a is of infinite order so that $a \otimes e$ is of infinite order in $A \otimes B$. Now $p_n : B \to Z_n$ projects e onto e_n and $a \otimes e_n = a/n \otimes ne_n = 0$. Thus $p'_n : A \otimes B \to A \otimes Z_n$ annihilates $A \otimes B$ so that the induced map $p' : A \otimes B \to \to \prod_n A \otimes Z_n$ is certainly not (1—1).

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Some Remarks on Totally Disconnected ¹Sections of Monotone Open Mappings

by

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Let (X, f, Y) denote a triple such that X and Y are compact metric continua and f is a monotone open mapping (continuous transformation) of X onto Y such that, for each $y \in Y$, $f^{-1}(y)$ is non-degenerate. The point set K is said to be a *section* of (X, f, Y) if K is a closed totally disconnected subset of X and $f(K) = Y$ *). We consider the question of conditions on X or Y sufficient to guarantee the existence of a section of (X, f, Y). In 1942 ([1], section 7), Kelley proved the principal theorem about this question (our Theorem I) **). We shall note rather obvious sufficient conditions on X in Theorems II and III. Theorems IV and V assert the existence of examples showing that Theorems I and II are about as strong as possible.

THEOREM I. *(Kelley): If Y is finite dimensional, (X, f, Y) admits a section.*

This result is proved in section 7 of [1].

THEOREM II. *If X is one-dimensional, (X, f, Y) admits a section.*

Proof. From f open with only non-degenerate inverses it follows that there must be a positive lower bound δ on the diameters of the inverses of the points of Y. For each $p \in X$, let $U(p)$ be a neighbourhood of p with diameter $U(p) < \delta$ and the boundary of $U(p)$ totally disconnected. There is a finite set of such neighbourhoods covering X. The sum of their boundaries is a section of (X, f, Y).

THEOREM III. *If X is a subset of a 2-manifold, (X, f, Y) admits a section.*

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*) Note that a "section" is not a "cross-section" in the usual sense. Over K, f need not be — and if Y is non-degenerate cannot be — a homeomorphism.

**) More recently Kosiński [5] has shown an almost equivalent result.

Proof. Y must be a subset of a generalized cactoid [2] and hence must be at most two dimensional. Then Theorem III follows from Theorem I.

THEOREM IV. *For any* $n \geqslant 2$, *there exists a triple* (X, f, Y) *such that* X *is* n-*dimensional and* (X, f, Y) *does not admit a section.*

Proof. Kelley (sections 7 and 8 of [1]) shows that if ε is any positive number and H is any hereditarily indecomposable continuum, then H admits a monotone open mapping g onto a space Z such that, for each $z \in Z$, $0 < \text{diameter } g^{-1}(z) < \varepsilon$. He further shows in his argument that if, for each $\varepsilon > 0$, there is such a mapping g_ε of H with $0 < \text{diameter } g_\varepsilon^{-1}(z) < \varepsilon$, for each $z \in g_\varepsilon(H)$, such that $(H, g_\varepsilon, g_\varepsilon(H))$ admits a section, then dim $H = 1$. But Bing [3] has shown that, for any $n > 1$, there exists a hereditarily indecomposable continuum X of dimension n with X imbedded in $(n+1)$--space. Then from Kelley's results there exists a monotone open mapping f of X with non-degenerate inverses such that $(X, f, f(X))$ admits no section.

THEOREM V. *For any* $n > 3$ *) and each manifold* X *with* dim $X = n$, *there exists a monotone open mapping* f *of* X *such that* $(X, f, f(X))$ *admits no section.*

Proof. This result follows from Theorem IV and the principal theorem of [4] which asserts (among other things) that if g is any monotone open mapping of a compact subset Z of X with dim $Z \leqslant (n-2)$ then there exists a monotone open mapping f of X such that f on Z is g and $f(X - Z)$ does not intersect $f(Z)$.

COROLLARY. *There exists a triple* (X, f, Y) *with* X *(and hence* Y) *locally connected and with* (X, f, Y) *not admitting a section.*

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*) Theorem V is probably true for $n = 3$ also, but the techniques of [4] need careful specialization with respect to Bing's two-dimensional example to show this.

On Connections Between the Homology Properties of a Set and of Its Frontier

by

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1. r-homomorphisms. By an *r-homomorphism* will be meant a homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ of a group $\mathfrak{A}$ into a group $\mathfrak{B}$ such that there exists a homomorphism $\psi : \mathfrak{B} \to \mathfrak{A}$ satisfying

$$(1) \qquad\qquad \varphi\psi(y) = y \qquad \text{for every} \quad y \in \mathfrak{B}.$$

Obviously, ψ maps $\mathfrak{B}$ isomorphically onto some subgroup of $\mathfrak{A}$.

By a *retracting homomorphism* will be meant an endomorphism $\varphi : \mathfrak{A} \to \mathfrak{A}$ satisfying $\varphi\varphi(x) = x$ for every $x \in \mathfrak{A}$. Every retracting homomorphism is an r-homomorphism from $\mathfrak{A}$ to $\varphi(\mathfrak{A})$.

It is easily seen that

(2) If $\varphi_1 : \mathfrak{A} \to \mathfrak{B}$ and $\varphi_2 : \mathfrak{B} \to \mathfrak{C}$ are r-homomorphisms, then so also is $\varphi_2\varphi_1 : \mathfrak{A} \to \mathfrak{C}$.

(3) r-homomorphisms are identical with transformations of the form $\varphi_2\varphi_1$, where φ_1 is a retracting homomorphism and φ_2 is an isomorphism.

(4) in order that $\varphi : \mathfrak{A} \to \mathfrak{B}$ be an r-homomorphism it is necessary and sufficient that $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$, where $\mathfrak{A}_1 = \varphi^{-1}(0)$ and $\varphi : \mathfrak{A}_2 \to \mathfrak{B}$ is an isomorphism onto.

(2) and (3) and the sufficiency condition of (4) are trivial. For the necessity proof see e. g. [1], p. 366, Theorem 7.7.

(5) Let $\varphi_0 : \mathfrak{A} \approx \mathfrak{B} + \mathfrak{B}'$ be of the form $\varphi_0(x) = (\varphi(x), \varphi'(x))$, $\varphi(x) \in \mathfrak{B}$, $\varphi'(x) \in \mathfrak{B}'$. Then $\varphi : \mathfrak{A} \to \mathfrak{B}$ is an r-homomorphism.

For the homomorphism $\psi : \mathfrak{B} \to \mathfrak{A}$ defined by $\psi(y) = \varphi_0^{-1}(y, 0)$ for every $y \in \mathfrak{B}$ satisfies (1).

(6) If $\varphi_1 : \mathfrak{A} \to \mathfrak{B}$ and $\varphi_1 : \mathfrak{B} \to \mathfrak{C}$ are homomorphisms and $\varphi_2\varphi_1$ is an r-homomorphism, then φ_2 is an r-homomorphism.

For let $\psi: \mathfrak{C} \to \mathfrak{A}$ be such that $\varphi_2 \varphi_1 \psi(y) = y$ for every $y \in \mathfrak{C}$. Define $\psi_2: \mathfrak{C} \to \mathfrak{B}$ by $\psi_2(y) = \varphi_1 \psi(y)$. Then $\varphi_2 \psi_2(y) = \varphi_2 \varphi_1 \psi(y) = y$.

2. Homology of a set and of its frontier: homotopy assumptions. "Space" will stand for Hausdorff space, "compact" will stand for bi-compact, "mapping" will always denote a continuous mapping. Let a compact coefficient group be chosen. $H_k(A)$ will denote the Čech homology group of the space A based on this coefficient group. The 0-dimensional homology group may be reduced or not. If $f: X \to Y$ is a mapping, then $f_*: H_k(X) \to H_k(Y)$ will denote the homomorphism induced by f. The precise meaning of all these terms is explained in [2].

By a continuous deformation in the space M of the set $A \subset M$ into the set B will be meant a mapping $f: A \times \langle 0,1 \rangle \to M$ satisfying $f(x,0) = x$, $f(A,1) \subset B$.

THEOREM 1. *Let A be compact subset of a space M and suppose that there exists a continuous deformation $f(x,t)$ in M of A into $\overline{M-A}$. Let $F = \mathrm{Fr}\, A$ and $i: F \to A$ be the inclusion mapping.*

Then $i_*: H_k(F) \to H_k(A)$ is an r-homomorphism for every k.

Proof. Let C denote the space obtained from the product $A \times \langle 0,2 \rangle$ by identification of the set $A \times (2)$ to a point. C is a cone with base $A \times (0)$. We shall identify the base with A. Let us consider the following subsets of C:

$$T(A) = \mathop{E}_{(x,t)} [x \in A, \ 0 \leqslant t \leqslant 1; \ f(x,t) \in A],$$

$$T(F) = \mathop{E}_{(x,t)} [x \in A, \ 0 \leqslant t \leqslant 1; \ f(x,t) \in F],$$

$$D = \big(C - T(A)\big) \cup T(F).$$

Obviously, the triad $(C; T(A), D)$ is compact and

$$T(A) \cap D = T(F), \ T(A) \cup D = C.$$

Now, in the Čech theory every compact triad is a proper triad ([2], p. 257). Therefore the Mayer-Vietoris sequence

$$\ldots \to H_{k+1}(C) \to H_k\big(T(F)\big) \xrightarrow{\varphi} H_k\big(T(A)\big) + H_k(D) \to H_k(C) \to \ldots$$

of the triad $\big(C; T(A), D\big)$ is exact ([3], p. 39). But C is contractible so that $\varphi: H_k\big(T(F)\big) \approx H_k\big(T(A)\big) + H_k(D)$. Since $\varphi(x) = \big(i_{1*}(x), -i_{2*}(x)\big)$, where $i_1: T(F) \to T(A)$ and $i_2: T(F) \to D$ are inclusions, it follows from (5) that $i_{1*}: H_k\big(T(F)\big) \to H_k(T(A))$ is an r-homomorphism.

Let us observe now that the mapping r defined for every $(x,t) \in T(A)$ by $r(x,t) = f(x,t)$ maps $T(A)$ in A and $T(F)$ in F and is a retraction of

$T(A)$ onto A. Therefore the diagram

$$\begin{array}{ccc} H_k\big(T(F)\big) & \xrightarrow{\ i_{1*}\ } & H_k\big(T(A)\big) \\ \downarrow{r_*} & & \downarrow{r_*} \\ H_k(F) & \xrightarrow{\ i_*\ } & H_k(A) \end{array}$$

is commutative and the right vertical homomorphism is an r-homomorphism. But we have proved that i_{1*} is an r-homomorphism. Therefore, by commutativity, $i_* r_*$ is an r-homomorphism and the theorem follows from (6).

3. Homology of a set and of its frontier: homological assumptions.

THEOREM 2. *Let A and Z be compact subsets of a space M. Let $A \subset Z \subset M$ and denote $F = \operatorname{Fr} A$; $j_1 : A \to Z$ and $i : F \to A$ will denote inclusion mappings.*

a) *If $j_{1*} : H_k(A) \to H_k(Z)$ is trivial then $i_* : H_k(F) \to H_k(A)$ is onto.*

b) *If $H_k(Z) = 0 = H_{k+1}(Z)$ then $i_* : H_k(F) \to H_k(A)$ is an r-homomorphism.*

Proof. Denote $B = F \cup (Z - A)$ and let $j_2 : B \to Z$ be the inclusion mapping. Then $Z = A \cup B$, $F = A \cap B$ and the triad $(Z; A, B)$ is compact. Therefore the Mayer-Vietoris sequence

$$... \to H_{k+1}(Z) \to H_k(F) \xrightarrow{\varphi} H_k(A) + H_k(B) \xrightarrow{\psi} H_k(Z) \to ... ,$$

where $\psi(v_1, v_2) = j_{1*} v_1 + j_{2*} v_2$, is exact.

Now, if j_{1*} is trivial then $H_k(A) \subset \psi^{-1}(0) = \varphi\big(H_k(F)\big)$ and this implies that i_* is onto.

If $H_{k+1}(Z) = 0 = H_k(Z)$ then exactness and (5) yield the theorem.

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On Mappings which Satisfy Certain Conditions on Boundary

by

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1. This note is concerned with mappings of the form $f\colon (A, \operatorname{Fr} A) \to \to (B, \operatorname{Fr} B)$, where B is a compact subset of a compact space *) M with certain homology groups vanishing, and certain conditions are put upon $f_0 = f | \operatorname{Fr} A$. It is proved in **2** that if the homomorphism induced by f_0 on $H_k(\operatorname{Fr} A)$ is onto (resp. r-homomorphism **)) then so also is the homomorphism induced on $H_k(A)$ by f, provided $H_k(M) = 0 = H_{k+1}(M)$. (Actually, the theorem is stronger in that $\operatorname{Fr} A$ is replaced by an arbitrary compact subset of A).

The natural question arises: what can be said about $f_*\colon H_k(A) \to H_k(B)$ when it is assumed that $f_{0*}\colon H_k(\operatorname{Fr} A) \to H_k(\operatorname{Fr} B)$ is an isomorphism. A partial result is proved in **3**, but the main problem as to whether f_* is an isomorphism, provided A, B are compact subsets of S_n, and $f(\operatorname{Int} A) = \operatorname{Int} B$, remains unsolved.

However, if f_0 is a homeomorphism and if $f(\operatorname{Int} A) = \operatorname{Int} B$, then the question can be answered in the affirmative. This is done in **5**; the proof depends very much on Sitnikov's results [3]. In **6** some corollaries concerning the homology properties of the complement of B are listed.

I should like to express my gratitude to Professor K. Kuratowski who initiated these investigations and kindly informed me about problems which arose. In particular, Theorems 4 and 5 were proposed by him.

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*) Unless otherwise explicitly stated, the following conventions hold: By a space will be meant a Hausdorff space, compact will stand for bicompact, and mapping will denote continuous mapping. $H_k(A)$ will denote the Čech homology group of a space A over a compact group of coefficients. This group has been chosen once and for all. 0-dimensional homology groups will always be reduced (see [1] p. 18; this book will serve as the standard reference in homology theory).

We use Čech theory with compact coefficients since we need an exact and continuous homology theory. Each theorem proved in this note can be translated into a corresponding dual theorem in cohomology theory. These dual theorems hold with an arbitrary coefficient group in the category of R-modules (over a ring R with a unit element),

**) *i. e.* has the right inverse. See [2].

2. THEOREM 1. *Let M be a compact space acyclic* *) *in dimensions k, $k+1$. Let (A, A_0), $(B, \operatorname{Fr} B)$ be compact pairs with $B \subset M$. Let $f \colon (A, A_0) \to (B, \operatorname{Fr} B)$ and define* ****) $f_0 \colon A_0 \to \operatorname{Fr} B$ *by* $f_0 = f | A_0$.

If $f_{0*} \colon H_k(A_0) \to H_k(\operatorname{Fr} B)$ is onto (resp. r-homomorphism), then $f_* \colon H_k(A) \to H_k(B)$ is onto (resp. r-homomorphism).

Proof. Let us consider the commutative diagram ([1], p. 15):

$$
\begin{array}{ccc}
H_k(A_0) & \xrightarrow{\;i'_*\;} & H_k(A) \\
\downarrow{\scriptstyle f_{0*}} & & \downarrow{\scriptstyle f_*} \\
H_k(\operatorname{Fr} B) & \xrightarrow{\;i_*\;} & H_k(B),
\end{array}
$$

where the horizontal homomorphisms are induced by inclusions. By Theorem 2. in [2] i_* is an r-homomorphism, and thus is onto. If f_{0*} is onto, then so also is $i_* f_{0*} = f_* i'_*$, so that f_* is onto. If f_{0*} is an r-homomorphism, then so also is $i_* f_{0*} = f_* i'_*$ (by (2) in [2]) so that f_* is an r-homomorphism (by (6) in [2]). This completes the proof.

3. THEOREM 2. *Let N, M be two compact spaces acyclic in dimensions k, $k+1$. Let $A \subset N$, $B \subset M$ and let $f \colon (A, \operatorname{Fr} A) \to (B, \operatorname{Fr} B)$, $g \colon (B, \operatorname{Fr} B) \to (A, \operatorname{Fr} A)$ be two mappings. Let $f_0 \colon \operatorname{Fr} A \to \operatorname{Fr} B$, $g_0 \colon \operatorname{Fr} B \to \operatorname{Fr} A$ be the mappings induced by f, g respectively.*

Suppose that $g_{0*} f_{0*} \colon H_k(\operatorname{Fr} A) \to H_k(\operatorname{Fr} A)$ is the identity mapping. Then $g_* \colon H_k(B) \to H_k(A)$ is the left inverse to $f_* \colon H_k(A) \to H_k(B)$, so that f_* is an isomorphism into.

Proof. Again, we have the diagram

$$
\begin{array}{ccc}
H_k(\operatorname{Fr} A) & \xrightarrow{\;i'_*\;} & H_k(A) \\
\downarrow{\scriptstyle f_{0*}} & & \downarrow{\scriptstyle f_*} \\
H_k(\operatorname{Fr} B) & \xrightarrow{\;i_*\;} & H_k(B),
\end{array}
$$

and a similar diagram for g. By Theorem 2 in [2] both i_* and i'_* have right inverses, j_* and j'_* respectively. Thus $i'_* j'_*(x) = x$, $x \in H_k(A)$ and $i_* j_*(y) = y$, $y \in H_k(B)$. Now, let $x \in H_k(A)$. By commutativity $i_* f_{0*} j'_*(x) = f_* i'_* j'_*(x)$, so that $i_* f_{0*} j'_*(x) = f_*(x)$ for every $x \in H_k(A)$. Similarly $i'_* g_{0*} j_*(y) = g_*(y)$ for every $y \in H_k(B)$. Therefore $g_* f_*(x) = i'_* g_{0*} j_* i_* f_{0*} j'_*(x) = x$ for every $x \in H_k(A)$. Hence f_* has the left inverse, thus being an isomorphism (into).

COROLLARY. *Let M, N, A, B, f, g and f_0, g_0 be as in Theorem 2. Suppose that $f_{0*} \colon H_k(\operatorname{Fr} A) \to H_k(\operatorname{Fr} B)$ is onto and that $g_{0*} f_{0*} \colon H_k(\operatorname{Fr} A) \to H_k(\operatorname{Fr} A)$ is the identity. Then $f_* \colon H_k(A) \to H_k(B)$ is an isomorphism onto.*

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*) *i. e.* the corresponding homology groups vanish.

**) the mapping $A \to B$ induced by f will also be denoted by f.

Proof. By Theorem 1 f_* is onto, by Theorem 2 it is an isomorphism.

4. Let U be an open subset of S_n. Denote by C the complex of a fixed locally finite triangulation of U. A subcomplex C_a of C will be called *counterfinite* if C_a contains all but a finite number of simplexes of C. We write $\alpha < \beta$ if $C_a \supset C_\beta$; homomorphism $\pi_a^\beta \colon H_k(C, C_\beta) \to H_k(C, C_a)$ is defined as induced by inclusion mapping $(C, C_\beta) \to (C, C_a)$. The system $\{H_k(C, C), \pi_a^\beta\}$ then forms an inverse system of groups *). The inverse limit of this system will be denoted by $H_k(C)$. (We recall that we use a compact group of coefficients. Comp. [1], p. 251-2. $H_k(C)$ is known as the homology group based on infinite chains).

LEMMA 1. *Let $A \subset S_n$ be compact and let C be a triangulation of* $\operatorname{Int} A$. *There is an isomomorphism* $i \colon H_k(C) \to H_k(A, \operatorname{Fr} A)$ *which is natural in the following sense. If $B \subset S_n$ is compact,* $f \colon (A, \operatorname{Fr} A) \to (B, \operatorname{Fr} B)$, $f(\operatorname{Int} A) = {} = \operatorname{Int} B$ *and f is simplicial with respect to the two given triangulations C and C' of* $\operatorname{Int} A$, $\operatorname{Int} B$ *respectively, then there is commutativity in the diagram*

$$
\begin{array}{ccc}
H_k(A, \operatorname{Fr} A) & \xrightarrow{\;f_*\;} & H_k(B, \operatorname{Fr} B) \\
\Big\uparrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle i} \\
H_k(C) & \xrightarrow{\;f_*\;} & H_k(C').
\end{array}
$$

Proof. C_a' will be the system of all counterfinite subcomplexes of C', $C_a = f^{-1} C_a'$. It is easily seen that C is counterfinite and that $\lim H_k(C, C_a) = H_k(C)$. We shall denote by D_a (D_a') the subcomplex of C (C') consisting of all those simplexes which are not in C_a' (C_a') and of their faces. For every complex D, $|D|$ will denote the underlying space.

Consider the following two diagrams:

$$
\begin{array}{ccc}
A, |C_a| \cup \operatorname{Fr} A) & \xrightarrow{f_*} & H_k(B, |C_a'| \cup \operatorname{Fr} B) \\
\uparrow{\scriptstyle i_3} & & \uparrow{\scriptstyle i_3} \\
(|D_a|, \operatorname{Fr} |D_a|) & \xrightarrow{f_*} & H_k(|D_a'|, \operatorname{Fr} |D_a'|) \\
\uparrow{\scriptstyle i_2} & & \uparrow{\scriptstyle i_2} \\
k(D_a, D_a \cap C_a) & \xrightarrow{f_*} & H_k(D_a', D_a' \cap C_a') \\
\uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_1} \\
H_k(C, C_a) & \xrightarrow{f_*} & H_k(C', C_a')
\end{array}
\qquad
\begin{array}{ccc}
H_k(\lim(A, |C_a| \cup \operatorname{Fr} A)) & \xrightarrow{f_*} & H_k(\lim(B, |C_a'| \cup \operatorname{Fr} B)) \\
\uparrow{\scriptstyle i_5} & & \uparrow{\scriptstyle i_5} \\
\lim H_k(A, |C_a| \cup \operatorname{Fr} A) & \xrightarrow{f_*} & \lim H_k(B, |C_a'| \cup \operatorname{Fr} B) \\
\uparrow{\scriptstyle i_4} & & \uparrow{\scriptstyle i_4} \\
\lim H_k(C, C_a) & \xrightarrow{f_*} & \lim H_k(C', C_a').
\end{array}
$$

In the left hand diagram i_1 and i_3 are isomorphisms induced by inclusions and i_2 is the isomorphism from the Uniqueness Theorem ([1], p. 100). Thus $i_3 i_2 i_1$ is an isomorphism and commutativity holds in all squares.

---

*) In the sequel we shall not show the projections. In each case it will be clear how they are defined. (For all details concerning inverse systems and limit groups see [1], Ch. VIII and X).

This implies that $i_3 i_2 i_1$ maps isomorphically the inverse system $\{H_k(C, C_a)\}$ onto $\{H_k(A, |C_a| \cup \operatorname{Fr} A)\}$. Thus putting $i_4 = \lim i_3 i_2 i_1$ we infer that the commutativity holds in the lower right-hand square, where lim denotes inverse limit of the appropriate inverse system. But the upper square is commutative by Theorem 2.1 in [1], p. 259 and i_5 is an isomorphism onto in view of the continuity of the Čech theory ([1], p. 261). Since $\lim(A, |C_a| \cup \operatorname{Fr} A) = (A, \operatorname{Fr} A)$ and similarly for $(B, \operatorname{Fr} B)$, the lemma follows (with $i = i_5 i_4$).

5. Let U be an open subset of S_n. Let $f\colon U \to S_n$. Then f defines the upper semicontinuous decomposition of U into closed sets $F_y = f^{-1}(y)$, $y \in f(U)$. f will be said to be a damping map if $\delta(F_y) \to 0$ with $\varrho(F_y, \operatorname{Fr} U) \to 0$. This term was introduced by Sitnikov in [3].

THEOREM 3. *Let A, B be compact subsets of S_n and let $f\colon (A, \operatorname{Fr} A) \to (B, \operatorname{Fr} B)$. Define $f_0\colon \operatorname{Fr} A \to \operatorname{Fr} B$ and $f_1\colon \operatorname{Int} A \to B$ by $f_0 = f | \operatorname{Fr} A$, $f_1 = f | \operatorname{Int} A$. Suppose that f_1 is damping and that $f_1(\operatorname{Int} A) \supset \operatorname{Int} B$.*

If $f_{0*}\colon H_k(\operatorname{Fr} A) \to H_k(\operatorname{Fr} B)$ is an isomorphism onto for each k; then f induces the isomorphism of the homology sequence (see [1], p. 15) of $(A, \operatorname{Fr} A)$ onto that of $(B, \operatorname{Fr} B)$.

Proof. We shall use three results of Sitnikov [3]:

a) $f_1(\operatorname{Int} A)$ is open in S_n,

b) there exist triangulations C, C' of $\operatorname{Int} A$, $f(\operatorname{Int} B)$ and such a simplicial approximation g_1 of f_1 with respect to C, C' that $\varrho(f_1(x), g_1(x)) \to 0$ with $x \to \operatorname{Fr} A$,

c) $g_{1*}\colon H_k(C) \to H_k(C')$ is an isomorphism onto for each k.

Now, it follows from a) that $f_1(\operatorname{Int} A) \subset \operatorname{Int} B$, therefore $f_1(\operatorname{Int} A) = \operatorname{Int} B$. It follows from b) that g_1 may be extended to a mapping $g\colon A \to B$ such that $f | \operatorname{Fr} A = g | \operatorname{Fr} A$. Therefore we are in a position to apply lemma 1 and it follows, by c), that $g_*\colon H_k(A, \operatorname{Fr} A) \to H_k(B, \operatorname{Fr} B)$ is an isomorphism onto. But g and f are homotopic rel. $\operatorname{Fr} A$, so that f_* and g_* coincide. The theorem itself follows then from the "five" lemma ([1], p. 16).

As may be shown on simple examples $f | \operatorname{Fr} A$ need not be neither a homeomorphism nor a mapping onto the set $\operatorname{Fr} B$. In connection with this remark is the following

THEOREM 4. *Let A, B be compact subsets of S_n and let $f\colon (A, \operatorname{Fr} A) \to (B, \operatorname{Fr} B)$. Define f_0, f_1 as in Theorem 3 and suppose that $f_1(\operatorname{Int} A) = \operatorname{Int} B$ and that f_0 maps $\operatorname{Fr} A$ homeomorphically onto $\operatorname{Fr} B$.*

Then f induces the isomorphism of the homology sequence of $(A, \operatorname{Fr} A)$ onto that of $(B, \operatorname{Fr} B)$.

Proof. In order to apply Theorem 3 we have only to prove that f_1 is damping. Let $y_n \in \operatorname{Int} B$ and let $F_n = f^{-1}(y_n)$ be so chosen that

$\lim \varrho(F_n, \operatorname{Fr} A) = 0$. Suppose there exist points $x_n, x_n' \in F_n$ such that $\varrho(x_n, x_n') \geqslant a > 0$ and $\varrho(x_n, \operatorname{Fr} A) \to 0$. Let $x = \lim x_n$, $x' = \lim x_n'$. Then $x \in \operatorname{Fr} A$, $f(x) = f(x')$ and $x \neq x'$. Since $f(\operatorname{Int} A) \cap f(\operatorname{Fr} A) = 0$ we infer that $x' \in \operatorname{Fr} A$. This contradicts the fact that f is a homeomorphism on $\operatorname{Fr} A$. The proof is thus complete.

6. LEMMA 2. *Let $\mathfrak{A}_i$, $i = 1, 2$ be discrete groups and let $\mathfrak{B}_i$ be the character group of $\mathfrak{A}_i$, $i = 1, 2$. Let $\varphi\colon \mathfrak{B}_1 \to \mathfrak{B}_2$ be a homomorphism and let, $\overline{\varphi}\colon \mathfrak{A}_2 \to \mathfrak{A}_1$ be the homomorphism dual to φ (see [4], p. 691).*

a) *If φ is onto, then $\overline{\varphi}$ is an isomorphism into;*

b) *if φ has the right inverse ψ, then $\overline{\varphi}$ is the right inverse to the dual to ψ.*

Proof. If $a \in \mathfrak{A}_2$ then a is a homomorphism of $\mathfrak{B}_2$ into the group of reals mod 1 and $\overline{\varphi}(a)$ is by definition the homomorphism of $\mathfrak{B}_1$ into the group of reals mod 1 defined by $a\varphi$. The first part of lemma a is known (see e. g. [4], 5.4. a). Now let $\psi\colon \mathfrak{B}_2 \to \mathfrak{B}_1$ be the right inverse to φ and $\overline{\psi}$ its dual. Then $\varphi\psi = 1$ and $\overline{\psi}(c) = c\psi$, $c \in \mathfrak{A}_1$. Therefore $\overline{\psi}\overline{\varphi}(a) = \overline{\psi}(a\varphi) = a\varphi\psi = a$ for every $a \in \mathfrak{A}_2$.

From now on M_n, N_n will stand for two n-dimensional compact and orientable n-dimensional manifolds. $\mathfrak{A}$ will be a countable discrete group and $\mathfrak{B}$ will be dual to $\mathfrak{A}$, i. e. the character group of $\mathfrak{A}$. We assume that $H_{n-k-1}(M_n, \mathfrak{A}) = H_{n-k}(M_n, \mathfrak{A}) = 0 = H_{n-k-1}(N_n, \mathfrak{A}) = H_{n-k}(N_n, \mathfrak{A})$ for a fixed number k, $0 \leqslant k \leqslant n - 1$.

THEOREM 5. *Let (A, A_0), $(B, \operatorname{Fr} B)$ be two compact pairs such that $A \subset N_n$, $B \subset M_n$. Let $f\colon (A, A_0) \to (B, \operatorname{Fr} B)$ and define $f_0\colon A_0 \to \operatorname{Fr} B$ by $f_0 = f | A_0$.*

If $f_{0*}\colon H_k(A_0, \mathfrak{B}) \to H_k(\operatorname{Fr} B, \mathfrak{B})$ is a homomorphism onto [resp. an r-homomorphism] then $H_{n-k-1}(M_n - B, \mathfrak{A})$ (singular homology group) is isomorphic with a subgroup [resp. a direct summand] of $H_{n-k-1}(N_n - A, \mathfrak{A})$.

Proof. It is known that our assumptions imply

a) $H_{k-1}(M_n, \mathfrak{B}) = 0$ provided $k < n - 1$, and $H_k(M_n, \mathfrak{B}) = 0$;

b) $H_{n-k-1}(N_n - A, \mathfrak{A})$ is dual to $H_k(A, \mathfrak{B})$ and
$H_{n-k-1}(M_n - B, \mathfrak{A})$ is dual to $H_k(B, \mathfrak{B})$.

(For both assertions see e. g. [4], p. 701).

Now, if $k < n - 1$ then a) holds and enables us to use Theorem 1. We infer that if f_{0*} is onto [resp. has the right inverse] then $f_*\colon H_k(A, \mathfrak{B}) \to H_k(B, \mathfrak{B})$ is onto [resp. has the right inverse]. Hence, by b) and lemma 2a [resp. lemma 2b] there exists an isomorphism of $H_{n-k-1}(M_n - B, \mathfrak{A})$ into $H_{n-k-1}(N_n - A, \mathfrak{A})$ and the theorem follows [resp. an r-homomorphism of $H_{n-k-1}(N_n - A, \mathfrak{A})$ onto $H_{n-k-1}(M_n - B, \mathfrak{A})$ and the theorem follows by [2] (4)].

If $k = n-1$ then we apply Theorem 1 with $M = M_n - \varDelta$ where $\varDelta$ is an open simplex of M_n disjoint with B. The preceding argument still holds.

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Über die Bewegungsgleichung des magnetischen Dipols

von

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Die Elementarteilchen haben im allgemeinen neben der eventuellen elektrischen Ladung auch ein magnetisches Moment. Daher betrachten wir die elektrischen Ladungen und magnetischen Momente als die Quellen des elektromagnetischen Feldes. Für den Physiker ist es nicht nur interessant, was für ein elektromagnetisches Feld die bewegten Ladungen und Dipole erzeugen, sondern auch, wie sich diese Teilchen, die ein magnetisches Moment und eine elektrische Ladung haben, in einem gegebenen elektromagnetischen Feld bewegen. Das die Bewegung der elektrischen Ladungen beschreibende Lorentz'sche Kraftgesetz ist schon seit langer Zeit bekannt, so dass die Bewegung der Teilchen, die elektrische Ladung haben, in einem äusseren Felde kein neues Problem darstellt. (Die Berücksichtigung der Wirkung des Eigenfeldes kann jedoch neue Probleme wohl hineinbringen). Bei der Bewegung des magnetischen Dipols im elektromagnetischen Felde ist die Lage indes nicht so einfach. Mit diesen Problemen haben sich mehrere Forscher beschäftigt [1], ihre Ergebnisse sind aber nicht ganz übereinstimmend.

Vor kurzem hat L. Infeld eine neue Methode angegeben [2], [5], mit der man die Bewegungsgleichungen der linearen Feldtheorien sehr einfach bestimmen kann. Das Wesen dieser Methode besteht in folgendem: Nehmen wir zu den Gleichungen des linearen Feldes (z. B. des elektromagnetischen — oder Mesonfeldes) auch noch die Gleichungen des Gravitationsfeldes hinzu, so bekommen wir ein nichtlineares Gleichungssystem, aus dem man Bewegungsgleichungen ableiten kann [3]. Gehen wir dann zu der speziellen Relativitätstheorie durch den Grenzübergang $k \to 0$, $g_{\mu\nu} \to \delta_{\mu\nu}$ über, so bekommen wir die Bewegungsgleichungen der linearen Feldtheorien. (k bedeutet hier die Gravitationskonstante). Die Rechnungen können wir sehr einfach durchführen, wenn wir das Punktteilchen durch eine δ-Funktion darstellen.

In dieser Arbeit wird die Bewegungsgleichung des magnetischen Dipols mit Hilfe der oben angeführten Infeld'schen Methode bestimmt.

Hier beschäftigen wir uns nur mit spinlosen Teilchen. (Wir hoffen, zu dem Problem der Bewegungsgleichung des Spin-Teilchens später, im Rahmen einer allgemeinen Theorie, zurückzukehren). Das Dipolmoment des Teilchens beschreiben wir durch einen antisymmetrischen Tensor $M_{\alpha\beta}$. Die raumartigen Elemente dieses Tensors entsprechen dem magnetischen Moment, die zeitartigen Elemente -- dem infolge der Bewegung erzeugten elektrischen Dipolmoment des Teilchens. Die Dipolmomentdichte eines Punktteilchens kann folgendermassen angeschrieben werden:

$$(1) \qquad m_{\alpha\beta} = \int M_{\alpha\beta}\, \delta_{(4)}\, d\tau,$$

wo

$$(2) \qquad \delta_{(4)} = \delta(x_1 - \xi_1)\, \delta(x_2 - \xi_2)\, \delta(x_3 - \xi_3)\, \delta\left(\frac{x_4}{ic} - \frac{\xi_4}{ic}\right)$$

die vierdimensionale Dirac'sche δ-Funktion ist. $\xi_\alpha(\tau)$ beschreibt die Weltlinie, τ bedeutet die Eigenzeit des Teilchens *).

Das Teilchen besitzt im Ruhsystem nur ein magnetisches Moment. Diese Sachlage kann in relativistischer Form folgenderweise ausgedrückt werden [4]:

$$(3) \qquad M_{\alpha\beta}\, u_\beta = 0.$$

Unter u_β verstehen wir die Vierergeschwindigkeit des Teilchens. Das durch das Teilchen erzeugte elektromagnetische Feld wird durch die Maxwell'schen Gleichungen beschrieben:

$$(4) \qquad \frac{\partial f_{\mu\nu}}{\partial x_\nu} = 4\pi\, \frac{\partial m_{\mu\nu}}{\partial x_\nu},$$

$$(5) \qquad \frac{\partial f_{\mu\nu}}{\partial x_\lambda} + \frac{\partial f_{\nu\lambda}}{\partial x_\mu} + \frac{\partial f_{\lambda\mu}}{\partial x_\nu} = 0,$$

wo $f_{\mu\nu}$ der elektromagnetische Feldtensor ist. Nach dem Gedankengang von L. Infeld können wir die Bewegungsgleichung des magnetischen Dipols in folgender Form anschreiben:

$$(6) \qquad \frac{d}{d\tau}(m u_\alpha) = -\frac{dt}{d\tau} \int \frac{\partial E_{\alpha\beta}}{\partial x_\beta}\, dv.$$

m ist die Ruhmasse des Teilchens, und unter $E_{\alpha\beta}$ verstehen wir den Energie-Impuls-Tensor des elektromagnetischen Feldes. Das Raumintegral muss über ein Volumen erstreckt werden, welches das Teilchen im Innern enthält. Den Energie-Impuls-Tensor $E_{\alpha\beta}$ kann man aus der, die obigen Feldgleichungen ergebenden, Lagrange-Funktion durch Variation nach

---

*) Die griechischen Indices durchlaufen die Werte 1, 2, 3, 4, die lateinischen Indices die Werte 1, 2, 3. $x_1 = ict$.

dem metrischen Tensor $g_{\mu\nu}$ erhalten [8]:

$$(7) \qquad E_{\alpha\beta} = \frac{1}{4\pi} \left(f_{\alpha\mu} f_{\beta\mu} - \frac{\delta_\alpha \beta}{4} f_{\mu\nu} f_{\mu\nu} \right) - \frac{1}{c^2} (u_\alpha m_{\beta\mu} + u_\beta m_{\alpha\mu}) f_\mu,$$

wo

$$(8) \qquad f_u = f_{\mu\nu} u_\nu$$

ein Viervektor ist.

Die Divergenz des Energie-Impuls-Tensors lautet bei Berücksichtigung der Feldgleichungen [4], [5], wie folgt:

$$(9) \qquad \frac{\partial E_{\alpha\beta}}{\partial x_\beta} = -\frac{1}{2} m_{\mu\nu} \frac{\partial f_{\mu\nu}}{\partial x_\alpha} - \frac{\partial}{\partial x_\nu} \left[f_{\alpha\mu} m_{\mu\nu} + \frac{1}{c^2} (u_\alpha m_{\nu\mu} + u_\nu m_{\alpha\mu}) f_\mu \right].$$

Setzen wir nun diesen Ausdruck in das Integral [9], [11] ein und schreiben die Divergenz in den Klammern als eine dreidimensionale Divergenz plus partielle Ableitung nach der Zeit an. Das Integral der raumartigen Divergenz verschwindet auf Grund des Gauss'schen Satzes und der Beziehung (1); die zurückbleibenden Integrale kann man mit Hilfe der δ-Funktion leicht ausrechnen. Diese Rechnungen werden wir in einem Koordinatensystem ausführen, in dessen Ursprung das Teilchen ruht. In diesem System sind $u_1 = u_2 = u_3 = 0$, $u_4 = ic$, $M_{\alpha4} = 0$, aber

$$\frac{du_k}{dt} \neq 0, \qquad \frac{dM_{\alpha4}}{d\tau} \neq 0.$$

In diesem Koordinatensystem lauten die Bewegungsgleichungen folgendermassen:

$$(10) \qquad \frac{d}{d\tau} (mu_\alpha) = \frac{1}{2} M_{\mu\nu} \frac{\partial f_{\mu\nu}}{\partial x_\alpha} + \frac{1}{ic} \frac{\partial}{\partial t} \left[f_{\alpha\mu} M_{\mu4} + \frac{1}{c^2} (u_\alpha M_{4\mu} + u_4 M_{\alpha\mu}) f_\mu \right].$$

Hier müssen die Grössen der rechten Seite der Gleichung in dem Raumpunkt, in dem sich das Teilchen befindet, genommen werden. Diese Gleichung können wir in folgender kovarianten Form anschreiben:

$$(11) \qquad \frac{d}{d\tau} (mu_\alpha) = \frac{1}{2} M_{\mu\nu} \frac{\partial f_{\mu\nu}}{\partial x_\alpha} - \frac{1}{c^2} f_{\alpha\mu} \frac{dM_{\mu\nu}}{d\tau} u_\nu -$$
$$- \frac{1}{c^4} u_\alpha \frac{dM_{\nu\mu}}{d\tau} u_\nu f_{\mu\lambda} u_\lambda + \frac{1}{c^2} \frac{d}{d\tau} (M_{\alpha\mu} f_{\mu\nu} u_\nu).$$

Es ist sehr leicht einzusehen, dass diese Gleichung im oben verwendeten Koordinatensystem mit der Gleichung (10) identisch ist. Da diese eine kovariante Gleichung ist, behält sie ihre Gültigkeit in jedem Lorentz'schen System.

Die Feldstärke $f_{\mu\nu}$ zerlegen wir in das äussere Feld $f_{\mu\nu}^{ex}$ und das Eigenfeld $f_{\mu\nu}^s$:

$$(12) \qquad f_{\mu\nu} = f_{\mu\nu}^{ex} + f_{\mu\nu}^s.$$

Bezeichnen wir die vom Eigenfeld stammende sog. Eigenkraft mit F_a^s. Hier betrachten wir nur die Bewegung eines magnetischen Dipols im äusseren Felde; die Eigenkraft F_a^s wird in einer folgenden Arbeit bestimmt werden. Aus der Beziehung (3) folgt:

$$(13) \qquad \dot{M}_{\alpha\beta} u_\beta = -M_{\alpha\beta} \dot{u}_\beta .$$

Der Punkt bedeutet hier die Ableitung nach der Eigenzeit. Bei Berücksichtigung der Gleichung (13) können wir die Bewegungsgleichungen in folgender endlichen Form anschreiben:

$$(14) \qquad \frac{d}{d\tau}(mu_a) = F_a^s + \frac{1}{2} M_{\mu\nu} \frac{\partial f_{\mu\nu}^{ex}}{\partial x_a} + \frac{1}{c^2} f_{a\mu}^{ex} M_{\mu\nu} \dot{u}_\nu - $$
$$- \frac{1}{c^4} u_a M_{\mu\nu} f_{\mu\lambda}^{ex} u_\lambda \dot{u}_\nu + \frac{1}{c^2} \frac{d}{d\tau}(M_{a\mu} f_{\mu\nu}^{ex} u_\nu).$$

Schliesslich wollen wir die Frage besprechen, ob die Ruhmasse des Teilchens konstant ist oder nicht. Multiplizieren wir zu diesem Zweck die Gleichung (14) durch die Virergeschwindigkeit u_a, so ergibt sich für die Ruhmasse die folgende Gleichung:

$$(15) \qquad \frac{dm}{d\tau} = -\frac{1}{c^2}\left\{ F_a^s u_a + \frac{d}{d\tau}\left(\frac{1}{2} M_{\mu\nu} f_{\mu\nu}^{ex}\right) - \frac{1}{2}\dot{M}_{\mu\nu}\left[f_{\mu\nu}^{ex} - \frac{1}{c^2}(u_\mu f_{\nu\lambda}^{ex} u_\lambda - u_\nu f_{\mu\lambda}^{ex} u_\lambda)\right]\right\}.$$

Aus dieser Gleichung kann man die Abhängigkeit der Ruhmasse von der Eigenzeit ablesen. Zu dieser Beziehung gelangen einige Forscher auf Grund anderer Untersuchungen [6]. Es lässt sich aus der Gleichung (15) ersehen, dass die Ruhmasse des Teilchens nicht konstant bleibt.

Wenn wir annehmen, dass das magnetische Moment des Teilchens mit einem Spin verknüpft ist, dann lässt sich $\dot{M}_{\mu\nu}$ bestimmen. Es ergibt sich, dass das Glied, welches $\dot{M}_{\mu\nu}$ enthält, verschwindet, und die Ruhmasse des Teilchens, abgesehen von der Eigenkraft, in folgender Form ausgedrückt werden kann:

$$(16) \qquad m = m_0 - \frac{1}{2c^2} M_{\mu\nu} f_{\mu\nu}^{ax}.$$

Die Tatsache, dass die Gleichung (15) mit den Ergebnissen anderer Arbeiten, in denen der Spin des Teilchens in Betracht gezogen wurde, übereinstimmt, zeigt, dass die Berücksichtigung des Spins in die Bewegungsgleichung eventuell nur ein Glied hereinbringen kann, das zu u_a senkrecht ist. Die Bewegungsgleichung (14) stimmt mit der für den magnetischen Dipol spezialisierten Gleichung von M. Mathisson überein, abgesehen von einem solchen zu u_a senkrechten Glied.

An dieser Stelle sei Professor L. Infeld der beste Dank für seine wertvolle Ratschläge und seine Hilfe ausgesprochen. Der Polnischen Aka-

demie der Wissenschaften danke ich für die Bereitwilligkeit, mit der sie mir im Rahmen einer Studienreise den Studienaufenthalt in Warschau ermöglicht hat.

INSTITUT FÜR THEORETISCHE PHYSIK DER ROLAND EÖTVÖS UNIVERSITÄT, BUDAPEST

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A Simple Derivation of the Equations of Motion in Classical Electrodynamics

by

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Presented by L. INFELD on March 30, 1956

1. There are two methods usually applied to derive the equations of motion (briefly: e. m.) in classical electrodynamics. These are: (i) Dirac's method [1], (ii) the Infeld-Wallace method [2]. Both of these methods involve rather tedious computations.

The scheme given below is, in a certain sense, a compromise between the methods mentioned. Its idea is close to a general method, recently developed by one of us [3], of deriving the e. m. in linear theories.

2. The action for the system "charged particle + field" is:

$$(1) \qquad W = -m^{\text{mech}} c \int_{\sigma_1}^{\sigma_2} ds + \frac{e}{c} \int_{\sigma_1}^{\sigma_2} \widetilde{A}_a(\xi) \, d\xi^a - \frac{1}{4\pi c} \int_{\sigma_1}^{\sigma_2} d_4 x \, \frac{1}{4} f_{a\beta}(x) f^{a\beta}(x),$$

where $\xi^a(s)$ — the world line of the particle $(ds^2 = d\xi_a d\xi^a)$, m^{mech} — its mechanical mass, $A_a(x)$ — the 4-potential vector, $f_{a\beta} = A_{\beta,a} - A_{a,\beta}$ — the field, and

$$(2) \qquad \widetilde{A}_a(\xi) \underset{\text{df}}{=} \lim_{\varepsilon \to 0} \int d_3 x \, \delta_3^\varepsilon(\vec{x} - \vec{\xi}) \, A_a(\vec{x}, \xi^0) = \int d_3 x \, \delta_3(\vec{x} - \vec{\xi}) \, A_a(\vec{x}, \xi^0).$$

The function $\delta_3^\varepsilon(\vec{x} - \vec{\xi})$ is a spherically symmetrical model of Dirac's δ-function. It is necessary to distinguish between $\widetilde{A}_a(\xi)$ and $A_a(\xi)$, the reason being that some quantities which are indefinite at a certain point vanish at that point (because of the symmetry of δ), providing the procedure of taking them at a point is understood in the sense as passing from A_a to $\widetilde{A}_a$ in (2).

From $\delta W = 0$ follow the Maxwell and Lorentz equations:

$$(3) \qquad f^{a\beta}(\vec{x}, x^0), \beta = 4\pi e \delta_3(\vec{x} - \vec{\xi}(x^0)) \frac{d\xi^a(x^0)}{dx^0},$$

$$(4) \qquad m^{\text{mech}} c \frac{d^2}{ds^2} \xi_a = \frac{e}{c} \widetilde{f}_{\beta a}(\xi) \frac{d\xi^\beta}{ds} \equiv \frac{e}{c} (\widetilde{A}_{a,\beta}(\xi) - \widetilde{A}_{\beta,a}(\xi)) \frac{d\xi^\beta}{ds}.$$

Equations (4) can also be obtained by applying the following general formulae of [3]:

$$(5) \qquad m^{\text{mech}} c \, \frac{d^2}{ds^2} \, \xi_\beta = \frac{dx^0}{ds} \int d_3 x \, T^a_{\beta,a},$$

where the 4-divergence of the energy-momentum tensor is to be obtained from (3). Thus, no matter which equation, (1) or (5), is taken as the starting point, we always come to Eq. (4) in which ξ is substituted into $A_{a,\beta}$ by means of the δ-function, as in (2).

To obtain the e. m. for a charged particle such $A_a(x)$ must first be found from (3), as are necessary to evaluate $\widetilde{A}_{a,\beta}(\xi)$ in (4). Taking into account the Lorentz condition, $A^a,a \, (x) = 0$, we get from (3):

$$(6) \qquad \square A^a(x) = 4\pi e \, \frac{d \xi^a(x^0)}{dx^0} \, \delta_3(\vec{x} - \vec{\xi}(x^0)).$$

The retarded Green's function for this equation is $\dfrac{1}{|\vec{x}|} \, \delta \left(t - \dfrac{|\vec{x}|}{c} \right)$.
Thus, writing the time explicitly, the following retarded solutions of (6) are obtained:

$$(7a) \qquad A^0 = -e \int\limits_{-\sim}^{+\sim} dt' \int d_3 x' \delta_1 \frac{\left(t - t' - \dfrac{|\vec{x} - \vec{x}'|}{c} \right)}{|\vec{x} - \vec{x}'|} \, \delta_3(\vec{x} - \vec{\xi}(t')),$$

$$(7b) \qquad A^a = -\frac{e}{c} \int\limits_{-\sim}^{+\sim} dt' \int d_3 x' \delta_1 \frac{\left(t - t' - \dfrac{|\vec{x} - \vec{x}'|}{c} \right)}{|\vec{x} - \vec{x}'|} \, \delta_3(\vec{x} - \xi(t')) \, \dot{\xi}^a(t').$$

Integrating with respect to x' and expanding into series in $1/c$ we get, according to [4]:

$$(8a) \qquad A^0 = -e \sum_{\iota=0}^{\sim} \frac{1}{c^\iota} \frac{(-)^\iota}{\iota!} \left(\frac{d}{dt} \right)^{\iota} |\vec{x} - \vec{\xi}(t)|^{\iota-1},$$

$$(8b) \qquad A^a = -\frac{e}{c} \sum_{\iota=0}^{\sim} \frac{1}{c^\iota} \frac{(-)^\iota}{\iota!} \left(\frac{d}{dt} \right)^{\iota} |\vec{x} - \vec{\xi}(t)|^{\iota-1} \, \dot{\xi}^a(t).$$

In a system where $\xi^a(t) = 0$, the first three equations of (4) reduce to $m^{\text{mech}} \ddot{\xi}^a = e(\widetilde{A}_{0,a} - \widetilde{A}_{a,0})$, and the fourth becomes an identity. Keeping in mind definition (2) we obtain (after performing an integration by parts in $\widetilde{A}_{0,m}$):

$$(9) \qquad m^{\text{mech}} \ddot{\xi}^a(t) = e \int d_3 x \delta_3(\vec{x} - \vec{\xi}(t)) \left[A^0,a(\vec{x},t) + \frac{1}{c} \frac{\partial}{\partial t} \, A^a(\vec{x},t) \right].$$

Substitution of (7) yields:

$$(10) \qquad m^{\text{mech}} \ddot{\xi}^a \doteq e^2 \int d_3 x \delta_3(\vec{x} - \vec{\xi}) \frac{x^a - \xi^a}{|\vec{x} - \vec{\xi}|^3} +$$

$$- \frac{e^2}{c^2} \int d_3 x \delta_3(\vec{x} - \vec{\xi}) \sum_{\iota=0}^{\infty} \frac{(-)^\iota}{c^\iota} \left[\frac{l+1}{(l+2)!} \left(\frac{d}{dt}\right)^{\iota+2} |\vec{x} - \vec{\xi}|^{l-1} (x^a - \xi^a) + \right.$$

$$\left. + \frac{1}{\iota!} \left(\frac{d}{dt}\right)^{l+1} |\vec{x} - \vec{\xi}|^{l-1} \dot{\xi}^a \right].$$

The term of order zero in $1/c$ vanishes on account of the spherical symmetry of δ_3.

3. To obtain the e. m. the right hand side of (10) must be valuated. We introduce the symbol $[x - \xi]^k$ for terms of the type: $\text{const} \, |\vec{x} - \vec{\xi}|^k$, $\text{const} \, |\vec{x} - \vec{\xi}|^{k-1} (x^a - \xi^a)$, $\text{const} \, |\vec{x} - \vec{\xi}|^{k-2} (x^a - \xi^a) \, (x^b - \xi^b), \ldots$

Let $[\overset{s}{\xi}]$ denote terms of the type $\text{const} \cdot \left(\frac{d}{dt}\right)^s \xi^a$. It can easily be seen that $\frac{d}{dt} [x - \xi]^k = [\overset{1}{\xi}] [x - \xi]^{k-1}$. Using this formula it can be proved by induction that

$$(11) \qquad \left(\frac{d}{dt}\right)^P [x - \xi]^k = \sum_{s=1}^{P} \sum_{a_1 + a_2 + \ldots a_s = P} [\overset{a_1}{\xi}][\overset{a_2}{\xi}]\ldots[\overset{a_s}{\xi}][x - \xi]^{k-s}.$$

Using (11), and noting that $\int d_3 x \delta_3(\vec{x} - \vec{\xi})[x - \xi]^k = 0$, $k > 0$, we have:

$$(12) \qquad \int d_3 x \delta_3(\vec{x} - \vec{\xi}) \left(\frac{d}{dt}\right)^{l+2} [x - \xi]^l =$$

$$= \sum_{s=\iota}^{e+2} \sum_{a_1 + a_2 + \ldots a_s = l+2} [\overset{a_1}{\xi}]\ldots[\overset{a_s}{\xi}] \int d_3 x [x - \xi]^{l-s} \delta_3(\vec{x} - \vec{\xi}) =$$

$$= [\overset{1}{\xi}]^{l+2} \int d_3 x [x - \xi]^{-2} \delta_3(\vec{x} - \vec{\xi}) + [\overset{1}{\xi}]^l [\overset{2}{\xi}] \int d_3 x [x - \xi]^{-1} \delta_3(\vec{x} - \vec{\xi}) +$$

$$+ ([\overset{1}{\xi}]^{l-1} [\overset{3}{\xi}] + [\overset{1}{\xi}]^{l-2} [\overset{2}{\xi}]^l) \int d_3 x [x - \xi]^0 \delta_3(\vec{x} - \vec{\xi}).$$

Now, both $(d/dt)^{l+2} |\vec{x} - \vec{\xi}|^{l-1} (x^a - \xi^a)$, $(d/dt)^{l+1} |\vec{x} - \vec{\xi}|^{l-1} \dot{\xi}^a$ are of the type $(d/dt)^{l+2} [x - \xi]^l$. Noting that in our system $[\overset{1}{\xi}] = \text{const.} \, \dot{\xi}^a = 0$, we deduce from (12) that all the terms on the right-hand side of (10) with $l > 2$ vanish. For $l = 2$ the only "dangerous" term in $\int d_3 x \delta_3(\vec{x} - \vec{\xi}) \, (d/dt)^4 [x - \xi]^2$ (this expression comes from $\int d_3 x \delta_3(\vec{x} - \vec{\xi}) \, (d/dt)^4 |\vec{x} - \vec{\xi}| (x^a - \xi^a)$, $\int d_3 x \delta_3(\vec{x} - \vec{\xi}) \, (d/dt)^3 |\vec{x} - \vec{\xi}| \dot{\xi}^a$) can be the term of the type

$[\overset{2}{\xi}]^2 \int d_3 x \delta_3 (\vec{x} - \vec{\xi})[x - \xi]^0$. However, it can be seen that $[\overset{2}{\xi}]^2[x - \xi]^0$ can consist only of terms of the type

$$\dot{\xi}^a \dot{\xi}^b \frac{x^b - \xi^b}{|\vec{x} - \vec{\xi}|}, \qquad \dot{\xi}^b \dot{\xi}^b \frac{x^a - \xi^a}{|\vec{x} - \vec{\xi}|}, \qquad \frac{(x^a - \xi^a)(x^b - \xi^b)(x^c - \xi^c)}{|\vec{x} - \vec{\xi}|^3} \dot{\xi}^b \dot{\xi}^c.$$

Integrals of these functions, multiplied by $\delta_3(\vec{x} - \vec{\xi})$, vanish on account of the symmetry of the δ-function. Therefore the term with $l=2$ in (10) yields no contribution. It is evident from (10) that the contribution of the term with $l=1$ is $+\dfrac{2}{3}\dfrac{e^2}{c^3}\overset{...}{\xi}{}^a$. The contribution of the term with $l=0$ has the form

$$\int d_3 x \delta_3 (\vec{x} - \vec{\xi}) \left(\frac{d}{dt}\right)^2 [x - \xi]^0 = [\overset{2}{\xi}] \int d_3 x \delta_3 (\vec{x} - \vec{\xi})[x - \xi]^{-1}$$

and comes from

$$-\frac{e^2}{c^2} \int d_3 x \delta_3 (\vec{x} - \vec{\xi}) \left[\frac{1}{2}\left(\frac{d}{dt}\right)^2 |\vec{x} - \vec{\xi}|^{-1}(x^a - \xi^a) + \frac{d}{dt}|\vec{x} - \vec{\xi}|^{-1}\dot{\xi}^a\right].$$

It can be seen that the terms $[\overset{2}{\xi}][x - \xi]^{-1}$ here involved are

$$-\frac{e^2}{c^2}\left(\frac{1}{2}\frac{\ddot{\xi}^a}{|\vec{x} - \vec{\xi}|} + \frac{1}{2}\frac{(x^a - \xi^a)(x^b - \xi^b)}{|\vec{x} - \vec{\xi}|^3}\ddot{\xi}^b\right).$$

Since, however,

$$\int d_3 x \delta_3 (\vec{x}) \frac{x^a x^b}{|\vec{x}|^3} = \frac{1}{3}\delta^{ab}\int d_3 x \frac{\delta_3(\vec{x})}{|\vec{x}|},$$

the total contribution to (10) from the term with $l=0$ is

$$-\frac{2}{3}\frac{e^2}{c^2}\int d_3 x \frac{\delta_3(\vec{x})}{|\vec{x}|}\ddot{\xi}^a.$$

We denote m^{field} the divergent field mass $\dfrac{2}{3}\dfrac{e^2}{c^2}\int \dfrac{\delta_3(\vec{x})}{|\vec{x}|}$. Putting, in accordance with the renormalisation procedure, $m^{\text{mech}} + m^{\text{field}} = m^{\text{exp}}$, where m^{exp} is the experimental mass, we can finally write (10) in the form:

(13) $$m^{\text{exp}}\overset{...}{\xi}{}^a = \frac{2}{3}\frac{e^2}{c^3}\overset{...}{\xi}{}^a.$$

This is a tensor equation and must hold in any co-ordinate system. It can thus be generalised into the covariant form:

(14) $$m^{\text{exp}} c \frac{d^2}{ds^2}\xi_a = \frac{2}{3}\frac{e^2}{c}\left(\frac{d^3}{ds^3}\xi_a + \frac{d^2\xi^\beta}{ds^2}\cdot\frac{d^2\xi_\beta}{ds^2}\cdot\frac{d\xi_a}{ds}\right).$$

Adding, on the right-hand side of (13), the external force

$$\frac{e}{c} f^{\text{ext}}_{\beta a}(\xi) \frac{d\xi^\beta}{ds},$$

gives the same result as that obtained by Dirac.

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On a Composite Model for the New Particles

by

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Presented by L. INFELD on April 19, 1956

Recently, Nishijima - Gell - Mann's rule for the systematization of new particles has achieved a great success. It accounts for various facts obtained from experiments with cosmic rays and with high energy accelerators. Nevertheless, it would be desirable from the theoretical standpoint to find a more profound meaning hidden behind this rule. This point is the purpose of the present paper.

It seems to me that the present state of the theory of new particles is very similar to that of atomic nuclei 25 years ago. At that time, we knew a beautiful relation between the spin and the mass number of the atomic nuclei. Namely, the spin of the nucleus is always integer if the mass number is even, whereas the former is always half integer if the latter is odd. But unfortunately we could not understand the profound significance of this even-odd rule. This fact, together with other mysterious properties of the atomic nuclei, for instance, the beta disintegration in which the conservation of energy seemed to be invalid, led us to the pessimistic viewpoint that the quantum theory would not be applicable in the domain of the atomic nucleus. However, the situation was entirely changed after the discovery of the neutron. Ivanenko and Heisenberg immediately proposed a new model for the atomic nuclei in which neutrons and protons are considered to be their constituents. By assuming that the neutron has the spin of one half, they explained the even-odd rule for the spins of atomic nuclei as the result of the addition law for the angular momenta of the constituents. Moreover, they could reduce all the mysterious properties of atomic nuclei to those of the neutron contained in them.

Supposing that a similar situation is realized at present, I propose a compound hypothesis for new unstable particles to account for Nishijima - Gell - Mann's rule. In our model, the new particles are considered to be composed of four kinds of fundamental particles in the true sense,

that is, nucleons, anti-nucleon, Λ^0, and anti-Λ^0. If we assume that Λ^0 has such properties as were assigned by Nishijima and Gell-Mann, we can easily get their even-odd rule for the composite particles as the result of the addition laws for the ordinary spin, the isotopic spin and the strangeness. In the next table, the models and the properties of the new particles are shown together with those of the fundamental particles in the true sense.

TABLE

Name	Model	Isotopic Spin	Strangeness	Ordinary Spin
$\mathfrak{N}$		1/2	0	1/2
$\overline{\mathfrak{N}}$		1/2	0	1/2
Λ		0	-1	1/2
$\overline{\Lambda}$		0	1	1/2
π	$\mathfrak{N}+\overline{\mathfrak{N}}$	1	0	0
$\theta(\tau)$	$\mathfrak{N}+\overline{\Lambda}$	1/2	1	0
$\overline{\theta}(\tau)$	$\overline{\mathfrak{N}}+\Lambda$	1/2	-1	0
Σ	$\mathfrak{N}+\overline{\mathfrak{N}}+\Lambda$	1	-1	1/2
Ξ	$\overline{\mathfrak{N}}+\Lambda+\Lambda$	1/2	-2	1/2

Here $\mathfrak{N}$ and $\overline{\mathfrak{N}}$ denote nucleon and antinucleon respectively, whereas Λ and $\overline{\Lambda}$ denote Λ^0 and anti-$\overline{\Lambda}^0$ respectively.

As long as internal structure is not concerned, our model for new particles is identical with that of Nishijima and Gell-Mann. However, it should be stressed that the curious properties of the new particles can be reduced to those of Λ^0, just like the mysterious properties of the atomic nuclei have been reduced to those of the neutron. Hence our theory contains fewer arbitrary elements than was the case for the original one of Nishijima and Gell-Mann.

Moreover, it would be a very striking fact that almost all the composite particles which are stable against the decay due to strong interactions can be identified with the new particles found in cosmic rays.

Finally, it should be remarked that there are still other arguments for the compound hypothesis for elementary particles. In spite of the great success achieved by the advent of Tomonaga-Schwinger's technique, it has recently become clear that we cannot avoid the internal inconsistency of the quantum field theory as long as the point model for elementary particles is used. Especially in the case of the π-meson, it is necessary to introduce a cut-off radius of the order of proton Compton wave length in order to get reasonable results.

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Raman Spectra of Sulphur Dioxide Solutions

by

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Presented by W. RUBINOWICZ on February 13, 1956

The variation of Raman spectra in accordance with solution density was investigated for mixtures of SO_2 with pyridine, benzene and carbon tetrachloride. The spectrum of the intermolecular compound $C_5H_5N.SO_2$ was identified. On the basis of the spectrum, certain proposals were advanced as to the structure of this compound.

Introduction

In recent years, there has been widespread investigation into the structure and absorption spectra of such intermolecular compounds as salts of generalised Lewis acids and organic bases. There has been relatively little work concerning the vibration spectra of such complexes.

This paper deals with the investigation of the Raman spectra of mixtures of pyridine with sulphur dioxide. It has been confirmed by physico-chemical methods that there arises, in this system, a complex $C_5H_5N.SO_2$ with a bond energy of the order of several kcal./mol. [1], [2], [3]. On the other hand, it is known that the spectra of both substances show considerable dependence on the solvents. Thorough investigations have been made of the influence of the hydrogen bond on the vibrational spectrum of pyridine, in a mixture with water [4], ethyl alcohol [5], acetic acid [6] and pyrrole [7]. Investigation of the vibrational spectra of the mixtures of pyridine with sulphur dioxide can be further enlightening as to the character of the intermolecular bonds and its influence on the structure of the molecules.

Experimental procedure

The Raman spectra were investigated for mixtures of SO_2 and C_5H_5N containing 90.4; 79.1; 76.2; 65.1; 51.9; 37.9 and 17.1$^0/_0$ mol. SO_2, and (for purposes of comparison) mixtures of SO_2 with CCl_4 and C_6H_6.

The reagents used for the mixture were thoroughly dehydrated, since even a small quantity of moisture produces a strong orange-yellow colouring of the solution. The composition of the solutions was determined by iodometric titration of SO_2 previously absorbed in KOH.

The spectra were investigated with a Huet A spectrograph having a dispersion of 17 Å/mm. at 4500 Å.

The frequencies of the SO_2 lines and of some stronger pyridine lines are, as observed, listed in Tables I-III.

Conclusions

Two types of interaction can be expected between molecules:

1) intermolecular bonds, the result of which is the creation of a stoichiometric compound having the possible structures
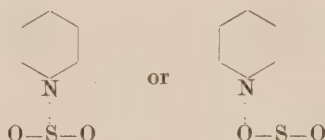
2) van-der-Waals interaction, primarily electrostatic, which plays a basic role here because of the large dipole moments of the simple molecules ($\mu_{SO_2} = 1.61$ D, $\mu_{C_5H_5N} = 2.2$ D) and the complex ($\mu \cong 6.2$ D [3]).

The first type of interaction leads to the simultaneous appearance, in the spectrum, of the lines corresponding to the vibrations of free molecules, and of modifications of these lines shifted from a few to several dozen cm.$^{-1}$, and differing from the primary lines in intensity and half width. The position of these lines does not depend, in the zero approximation, on the composition of the mixture. The "new lines" correspond to vibrations of the SO_2 and C_5H_5N molecules united in the complex, and therefore deformed as a result of intermolecular bonds.

The dependency of the intensity of the composition of the mixture permits each "new" line to be ascribed to a corresponding line of the free molecule. In Tables I and II, the lines of the complex are marked by an asterisk.

The van-der-Waals interaction, in general, causes a small shifting of the lines. This shifting varies continuously with the change in composition of the solution. In mixtures of sulphur dioxide with chemically inactive liquids (CCl_4, C_6H_6), where only the van-der-Waals effect can occur, a shifting of the SO_2 lines of the order of 1 to 3 cm.$^{-1}$ was observed. This is in agreement with the results of previous investigations [8]. The shift of lines due to free molecules in a mixture of SO_2 and C_5H_5N is also of the same order. The frequencies of vibration of the complex vary widely — from 3 to 12 cm.$^{-1}$.

TABLE I

Frequencies of the SO₂ line in mixtures of SO₂ and pyridine % mol. of SO₂

	100.0	90.4	79.1	76.2	68.1	51.9	37.9	17.1
ν_e (A_1)	524±2	~527±2	527±2	525±2	~527 (?)	~527 (?)	—	—
ν_2^*	—	~542 (?)	?	539±5	541±3	537±4	?	~535 (?)
ν_1^*	—	1130±4	1131±2	1132±2	1131.5±1	1133±2	1136±2	1137±2
ν_1 (A_1)	1145.5±0.5	1145.5±0.5	1145.5±0.5	1145.5±0.5	1145.5±0.5	—	—	—
ν_3^*	—	—	~1267	1272±10	1267±10	1285±10	?	1290±5
ν_3 (B_1)	1337±2	1336±2	1335±2	1334±4	1335±3	—	—	—

TABLE II

Frequencies of the pyridine line in mixtures of pyridine and SO₂ % mol. of pyridine

	100.0	82.9	72.1	48.1	31.9	23.8	20.9	9.6
ν_{6a} (A_1)	604±1	602±3	—	—	—	—	—	—
ν_{6a}^*	—	624±1	~630	623±1.5	627±1	628±2.5	628±1.5	627±3
ν_{6b} (B_1)	653±1	652±1	651.5±0.5	650±1	649.5±1	649.5±1	649±2	650±1
ν_1 (A_1)	991±1	992±1	992±1	992±1	—	—	—	—
ν_1^*	—	1005±2	1004±0.5	1004.5±0.5	1008±0.5	1008.5±0.5	1008±2	1008±1
ν_{12} (A_1)	1030.5±0.5	1030.5±0.5	1029.5±0.5	1029.5±1.5	1030±2	1030.5±2	1030±2	1030.5±1
ν_{18a} (A_1)	1066±2	1067.5±2	1067±2	1067±1	1066±2	1067±1	1067±2	1067±1
ν_{9a}^*	—	—	~1208 (?)	~1207 (?)	1208±2	1208±0.5	1207 (?)	—
r_{9a} (A_1)	1218±1	1217.5±0.5	1216±1	1212±2	—	—	—	—
ν_2 (A_1)	3055±2	3060±2	3061±2	3067±1	3075±5	3076±3	3076±2	3076±2

The frequency of vibration increases with the change from an equi-
molar solution to a solution in which the concentration of one of the
components is insignificant; this may be caused by the fact that the
interaction between molecules of the complex which have a particularly
high dipole moment, is considerably stronger than the remaining inter-
actions; this effect appears primarily in equimolar solutions where the
concentration of the complex is the highest.

TABLE III

SO$_2$ force constants and frequencies of vibration in different states

l_{S-O} Å	SO$_2$ gas 1.434	SO$_2$ liquid 1.45[12]	SO$_2$ complex 1.45
$\alpha\ S{<}^O_O$	119°30′	118°30′	115°
	in 10^5 dynes/cm.		
K_Q	9.92	9.69	9.15
$1/l^2\ K_\alpha$	0.79	0.81	0.90
$1/l\ A_{Q\alpha}$	0.195	0.195	0.155
H_{QQ}	0.019	0.019	0.019

Frequencies of vibration in cm.$^{-1}$.

	cal	obs[11]	cal	obs	cal	obs
ν_1	1154	1155	1147	1145.5	1132	1132
ν_2	518.5	518	522	523	541	539
ν_3	1361	1360	1337	1337	1290	1285

Disregarding the continuous shifts, referred to above, certain con-
clusions can, by using the mean values of the frequencies, be drawn as
regards the structure of the complex. Since the molecular interaction
energy is, approximately, considerably smaller than the bond energy of
the molecules themselves, the components of the complex can be studied
independently. The change in frequencies of vibration can be explained
by a change of the internal parameters of the molecules (geometry and
force constants) as a result of external disturbances. In the method used
by Volkenstein [9], the force constants of SO$_2$ molecules in the liquid
and in the complex were calculated on this basis. In this work, the force
constants taken as a starting point were those used by Polo and Wilson
for SO$_2$ gas molecules [10]. The constants were so chosen as to obtain
agreement with the frequencies observed, without introducing a correc-
tion for anharmonicity of vibrations. That also explains why the con-
stants from [10] were somewhat changed. The results of the calculation

are given in Table IV. In order to obtain experimental agreement, without drastic changes of the non-diagonal constants, it was necessary to assume a small change in the angle between the S-O bonds. This assumption is in agreement with the X.-ray measurements of the interatomic distances for liquid SO_2 [12].

TABLE IV

Force constants and frequencies of vibration of the SO_3 ion

$$l_{S-O}-1.39\ \text{Å} \quad \alpha\ S{<}^{O}_{O}-107.5°$$

Force constants in 10^5 dynes/cm.		Frequencies of vibration in cm.$^{-1}$.	
			cal.	obs.
K_Q	7.0	ν_1 (A) 960	966
$1/l^2\ K_\alpha$	1.11	ν_2 (A) 482	471
H_{QQ}	0.06	$\nu_{3,4}$ (E) 1134	—
$1/l\ A_{Q\alpha}$	0.075	$\nu_{5,6}$ (E) 609	612
$1/l^2\ l_{\alpha\alpha}$	−0.09		

Moreover, on the basis of the data given by Simon [13], the force constants for the $SO_3^=$ ion were estimated. Simon presents a somewhat different interpretation of the frequencies. The correctness of this interpretation seems doubtful. Simon in his calculations of the force constants uses a more simplified model. This leads to very high values of the non-diagonal constants and to somewhat different values of the diagonal force constants ($K_Q = 5.49$, $K_\alpha = 1.07$). However, even if we accept these constants, the final conclusions of this paper remain unchanged.

A comparison of the values of the force constants and angle shows that the parameters of the SO_2 molecule, disturbed by the intermolecular bond, take on values intermediate in relation to the free SO_2 molecule and the SO_3 ion. We can therefore conclude that the interaction with pyridine is an interaction in type (only considerably weaker), the same as the interaction with an oxygen ion, which leads to the stable bond (O_2S-0 bond) and to a change in the hybridisation of the sulphur electrons.

A comparison of the frequencies of free and bonded pyridine molecules, as observed by us, with the results obtained by several authors for the complex of pyridine with a hydrogen bond and for the pyridinium ion $(C_5H_5NH)^+$ [6], [7], [14], [15], indicates that the pyridine molecule has a structure intermediate as between a free molecule and a pyridinium ion.

Thus, it should be supposed that the complex has a transitional structure between

this was to be expected on the basis of Mulliken's theory of complexes of this type (intermolecular charge-transfer complex) [16].

This hypothesis requires further confirmation. Investigations of the spectra of sulphur dioxide ion mixtures with pyridine and other solvents are being continued.

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Reactions of Aromatic Amines with Cyanguanidine. Reactions of Phenyl Amidineurea and Their Derivatives with 2-aminopyridine

by

T. URBAŃSKI and B. SKOWROŃSKA-SERAFIN

Communicated by T. URBAŃSKI at the meeting of November 28, 1955

Continuing our former experiments [1], [2] on preparation and chemical properties of arylamidineureas, reactions of N_1-phenyl-N_2-amidineurea and N_1-(p-nitrophenyl)-N_2-amidineurea with 2-aminopyridine have now been studied. N_1-phenyl-N_2-amidineurea (I) nitrate on boiling with 2-aminopyridine gave guanidine and phenyl-2-pyridylurea (II) with a yield of 50%. No di-(2-pyridyl)-urea was, however, formed on prolonged boiling. N_1-(p-nitrophenyl)-N_2-amidineurea (III) hydrochloride reacted in a similar way giving at first p-nitrophenyl-2-pyridilurea (IV) (yield c. 66%). On prolonged boiling sym. dipyridylurea (V) and p-nitroaniline resulted:

(I) R= H
(III) R= NO$_2$

(II) R= H
(IV) R= NO$_2$

(V)

R= NO$_2$
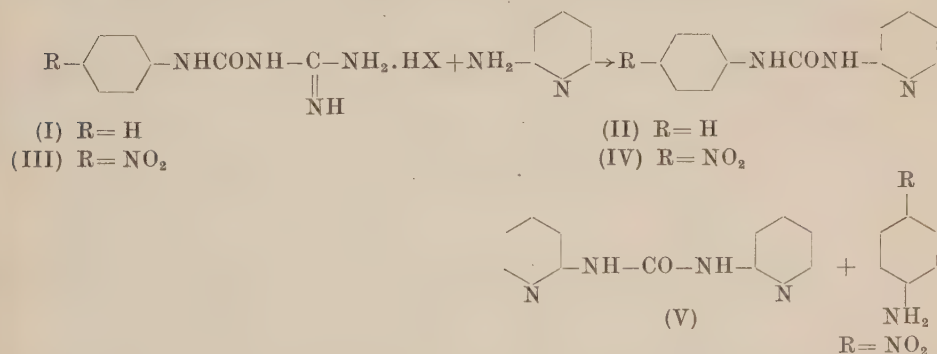
Thus, the trend of reactions of phenylamidineureas with 2-aminopyridine is similar to that with aniline [1], [2].

Experimental

1 g. hydrochloride of N_1-(p-nitrophenyl)-N_2-amidineurea (III) were mixed with 2 g. fused 2-aminopyridine and gently warmed for 1-2 minutes, to obtain a clear solution. On cooling, a precipitate of p-nitro-

phenyl-2-pyridylurea (IV) resulted. After crystallisation from butyl alcohol, colourless needles without a sharp m. p. were obtained (they sintered at 242° and sublimed at 247° as yellow needles). The yield of pure (IV) was 0.6 g. (66% theoretical).

Analysis:

$$C_{12}H_{10}O_3N_4 \quad \text{requires} \quad 55.8\% \text{ C; } 3.9\% \text{ H; } 21.7\% \text{ N.}$$
$$\text{found} \quad 55.6\% \text{ C; } 3.8\% \text{ H; } 21.4\% \text{ N.}$$

When (IV) was boiled further with 2-aminopyridine, N_1, N_2-(2-pyridyl)-urea (V) (m. p. 172-174°) and p-nitroaniline was formed. On prolonged boiling (e. g. 5 min.) of hydrochloride (II) with 2-aminopyridine, 0.3 g. p-nitroaniline (yield c. 53%) and a small quantity of (V) resulted.

Picrate of (IV) was formed from the alcoholic solution, m. p. 197-199° (decomp.).

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Reactions of Aromatic Amines with Cyanguanidine. Reactions of Phenylamidineurea and Their Derivatives with N-methylaniline
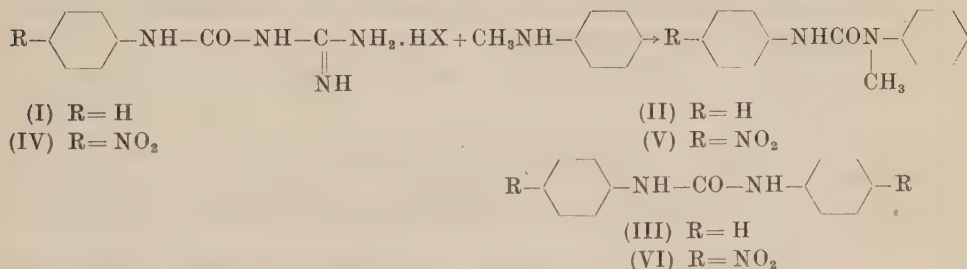
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Communicated by T. URBAŃSKI at the meeting of November 28, 1955

Continuing the former experiments on the preparation and chemical properties of aryl-amidineureas, reactions of N_1-phenyl-N_2-amidineurea and N_1-(p-nitrophenyl)-N_2-amidineurea with N-methylaniline have now been studied. Phenylamidineurea (I) nitrate, when boiled with N-methylaniline, gave guanidine and N-methylcarbanilide (II) with the yield of 65%. On prolonged boiling the yield of (II) was considerably reduced and carbanilide (III) and aniline could be isolated from the reaction-mixture.

The reaction of N_1-(p-nitrophenyl)-N_2-amidineurea (IV) hydrochloride with N-methylaniline gave guanidine, p-nitro-N-methylcarbanilide (V) (yield c. 26%) and a small amount of p, p'-dinitrocarbanilide (VI):

$$R-\langle\bigcirc\rangle-NH-CO-NH-\underset{\underset{NH}{\|}}{C}-NH_2 . HX + CH_3NH-\langle\bigcirc\rangle \rightarrow R-\langle\bigcirc\rangle-\underset{\underset{CH_3}{}}{N}HCON-\langle\bigcirc\rangle$$

(I) R= H   (II) R= H

(IV) R= NO_2 (V) R= NO_2

$$R-\langle\bigcirc\rangle-NH-CO-NH-\langle\bigcirc\rangle-R$$

(III) R= H

(VI) R= NO_2

These experiments indicate a certain difference between the behaviour of phenylamidineureas and its derivatives with primary and secondary aromatic amines, respectively.

The first stage of the reaction yielded in both instances urea derivatives, which contain the aryl from the starting amidineurea, attached to one of the nitrogen atoms of urea, and the radicals from the amine,

attached to another nitrogen atom of urea. Thus, compounds (II) and (V) are analogous with

$$R-\left\langle \right\rangle-NHCONH-\left\langle \right\rangle$$

which resulted from p - R - phenyl - amidine - ureas on boiling with aniline [1], [2].

In the second stage (prolonged boiling) the products are different: when primary amines were used, sym. diarylureas deriving from the amine resulted (hence carbanilide in the case of aniline was formed) [1], [2]; when secondary amine was used, sym. diarylurea, deriving from arylamidineurea resulted (compounds (III) and (VI)).

The mechanism of these reactions appears to be complicated; a chain reaction of free radicals similar to this, postulated previously by the authors [3], appears to be probable. Further experiments will be carried out.

Experimental

3 g. of phenylamidineurea (I) nitrate were mixed with 8 ml. of N-methylaniline and heated for a short time in an open flask to obtain a clear solution. The mixture was cooled and acidified with 15°/₀ HCl. The colourless precipitate was filtered or (better) extracted with benzene or ether. After drying with Na_2SO_4 the solvent was evaporated and the N-methylcarbanilide (II) crystallised in the form of colourless plates, m. p. 104 - 105°, yield 1.8 g. (c. 65°/₀ of theoretical).

N-methylcarbalinide (II) boiled for 30 minutes with aniline yielded carbanilide of m. p. 239 - 242° (II). Boiled with N - methylaniline for 4 hours, it remained unchanged.

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The Composition of Ternary Positive Azeotropes. I.

by

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Presented by W. ŚWIĘTOSŁAWSKI on March 6, 1956

1. Introductory remarks. In a previous paper [1], on the basis of the considerations resulting from Świętosławski's schema [2], [3] of the formation of positive azeotropes, the relation between the binary constants A_{12}, A_{23} and A_{13} of a system comprising three components 1, 2, and 3 was derived:

$$(1) \qquad A_{12} + A_{23} = A_{13}.$$

The constants A_{ij} are defined by the equations determining under isobaric conditions the activity coefficients f_i and f_j of binary regular solutions:

$$(2) \qquad RT \ln f_i = A_{ij} x_j^2.$$

From equation (1), a simple relation between the composition of the ternary and the binary azeotropes is now derived. It is assumed (as in the case of binary systems) that under isobaric equilibrium conditions the activity coefficients of the components 1, 2, and 3 in a ternary system are determined by the well-known equation, valid for regular solutions:

$$(3a) \qquad RT \ln f_1 = A_{12} x_2^2 + A_{13} x_3^2 + (A_{12} + A_{13} - A_{23}) x_2 x_3,$$

$$(3b) \qquad RT \ln f_2 = A_{12} x_1^2 + A_{23} x_3^2 + (A_{12} + A_{23} - A_{13}) x_1 x_3,$$

$$(3c) \qquad RT \ln f_3 = A_{13} x_1^2 + A_{23} x_2^2 + (A_{13} + A_{23} - A_{13}) x_1 x_2.$$

In the case of an azeotropic system under isobaric conditions, the application of relation (1) leads to the solution of several problems, particularly, to the relation between the compositions of the ternary azeotrope and those of binary azeotropes of a given system. Our treatment differs essentialy from that of Haase [4], who correlated the composition of a ternary azeotrope with binary constants A_{ij} and the vapour pres-

sures of pure components. It should be mentioned that the results obtained according to Haase, and those independently derived by Storonkin and Susarev [5], which may be called the "triangle cross-section rule", are generally in poor agreement with experimental data. Relation (1) and the latter considerations are restricted to positive azeotropic systems, in which, as has been mentioned before [1], the dispersion effect plays the main role. They are not applicable to systems in which there exist strong specific interactions between unlike molecules.

2. The activity coefficients. Substituting relation (1) into (3a), (3b) and (3c), we obtain:

$$\text{(4a)} \qquad RT \ln f_1 = A_{12}x_2^2 + A_{13}x_3^2 + 2A_{12}x_2x_3,$$

$$\text{(4b)} \qquad RT \ln f_2 = A_{12}x_1^2 + A_{23}x_3^2,$$

$$\text{(4c)} \qquad RT \ln f_3 = A_{13}x_1^2 + A_{23}x_2^2 + 2A_{23}x_1x_2.$$

Elimination of $x_2 = 1 - x_1 - x_3$ and of A_{13} according to relation (1) leads to the following equations:

$$\text{(5a)} \qquad RT \ln f_1 = A_{12}(1 - x_1)^2 + A_{23}x_3^2,$$

$$\text{(5b)} \qquad RT \ln f_2 = A_{12}x_1^2 + A_{23}x_3^2,$$

$$\text{(5c)} \qquad RT \ln f_3 = A_{12}x_1^2 + A_{23}(1 - x_3)^2.$$

These equations, determining the activity coefficients of the components in the ternary system, are basic for our further considerations.

Attention should be paid to the different form of equation (5b), compared with the symmetrical form of (5a) and (5c). This indicates that component 2 plays a specific role. This is the result of relation (1), according to which component 2 should be characterised by an intermediate internal pressure, since A_{13} is always greather than A_{12} and A_{23}.

3. Equations determining the azeotropic point. For determining the azeotropic point, we make use of the general thermodynamic relation

$$\text{(6)} \qquad p_i^0(T_{Az})f_i(T_{Az}, x_{Az}) = P \quad (i = 1, 2, 3),$$

which holds if the vapour obeys perfect gas laws. The vapour pressure of the pure component p_i^0 is given as a function of the temperature T by the equation

$$\text{(7)} \qquad \ln p_i^0 = \ln P + \frac{\Delta S_i}{R}\left(1 - \frac{T_i}{T}\right) \quad (i = 1, 2, 3),$$

in which P denotes as in equation (6) the external pressure, ΔS_i — the vaporisation entropy of component i at the boiling temperature T_i, under the pressure P.

Taking logarithms and multiplying equation (6) by RT_{Az}, and substituting into it $\ln p_i^0$ and $RT_{Az} \ln f_i$ according to equations (7) and (5), we obtain:

(8a) $$\Delta S_1(T_{Az}-T_1)+A_{12}(1-x_1)^2+A_{23}x_3^2=0,$$

(8b) $$\Delta S_2(T_{Az}-T_2)+A_{12}x_1^2+A_{23}x_3^2=0,$$

(8c) $$\Delta S_3(T_{Az}-T_3)+A_{12}x_1^2+A_{23}(1-x_3)^2=0.$$

This set of equations correlates the boiling temperature of the azeotrope, the mole fractions of the components in the azeotropic mixture, the boiling temperatures and the vaporisation entropies of the pure components in relation to the constants A_{12} and A_{23}. The solutions leading to several relations valid in ternary azeotropic systems are discussed below.

4. Composition of the ternary azeotrope. Assuming the equality of the vaporisation entropies $(\Delta S=\Delta S_1=\Delta S_2=\Delta S_3)$ of the components of a ternary system, we obtain, after subtracting equation (8b) from (8a) and (8c) respectively, the expressions:

(9) $$\Delta S(T_2-T_1)+A_{12}(1-2x_1^{(1,2,3)})=0,$$

(10) $$\Delta S(T_2-T_3)+A_{23}(1-2x_3^{(1,2,3)})=0,$$

where the index (1,2,3) denotes that the respective mole fractions represent the concentrations of the components 1, 2, and 3 in the ternary azeotrope. Dividing by ΔS and introducing the azeotropic range Z, [6], [7], as a parameter, we obtain the following equations:

(11) $$T_2-T_1+\tfrac{1}{2}Z_{12}(1-2x_1^{(1,2,3)})=0,$$

(12) $$T_2-T_3+\tfrac{1}{2}Z_{23}(1-2x_3^{(1,2,3)})=0,$$

and hence

(13) $$x_1^{(1,2,3)}=\frac{1}{2}+\frac{T_2-T_1}{Z_{12}},$$

(14) $$x_3^{(1,2,3)}=\frac{1}{2}+\frac{T_2-T_3}{Z_{23}}.$$

The compositions of binary azeotropes (1,2) and (2,3) are given by the following relations [6], [7]:

(15) $$x_1^{(1,2)}=\frac{1}{2}+\frac{T_2-T_1}{Z_{12}},$$

(16) $$x_3^{(2,3)}=\frac{1}{2}+\frac{T_2-T_3}{Z_{23}}.$$

Since the right sides of (13) and (15), and of (14) and (16) are identical, we obtain the simple relations

(17)
$$x_1^{(1,2,3)} = x_1^{(1,2)},$$

(18)
$$x_3^{(1,2,3)} = x_3^{(1,2)},$$

which may be formulated as a rule: "The mole fractions of the components characterised by extreme internal pressures (the largest and the smallest), are the same in the ternary azeotrope and in the two respective binary azeotropes, formed by them with the component having the medium value of the internal pressure".
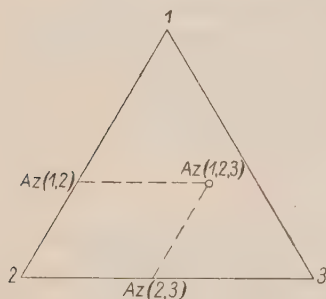

Fig. 1. Graphical presentation of the relation between the composition of the binary and the ternary azeotrope, resulting from equations (17) and (18)

It is shown in Fig. 1 that the composition of the ternary azeotrope is graphically represented as the crossing point of two lines, drawn through the azeotropic point (1, 2) parallel to axis 2, 3, and through point (2, 3) parallel to axis 1, 2. The rule resulting from our considerations may be called "rule of parallel lines". It is in contradiction with Storonkin's and Susarev's "triangle cross-section rule". It may be concluded from the schema, shown in Fig. 1, that the formation of three binary azeotropes *) is not a satisfactory condition for the formation of a ternary azeotrope. A necessary condition should be fulfilled additionally:

(19)
$$x_1^{(1,2)} + x_3^{(2,3)} < 1,$$

i. e., the respective parallel lines should cross each other within the triangle.

5. Comparison with the experimental results. Table I represents the compositions of binary and the respective ternary azeotropes found experimentally by Moraczewski [8], and those calculated according to (17) and (18). Although the data obtained were determined under isothermic conditions (at three different temperatures), nevertheless they seem to be close to those which would be found under constant pressure, because the boiling temperatures of the binary azeotropes do not differ significantly.

In the first column of Table I the components of the two ternary azeotropes formed by benzene (2), cyclohexane (1) and isopropanol (3)

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*) N o t e: If component 2 forms azeotropes with components 1 and 3, there exists an azeotrope (1, 3) according to the relation $Z_{12} + Z_{23} = Z_{13}$, previously derived [1].

<div align="center">TABLE I</div>

No.	System	$t_{eq.}$ °C.	$x_1^{(1,2)}$	$x_1^{(1,2,3)}$	$x_3^{(2,3)}$	$x_3^{(1,2,3)}$	$x_1^{(1,2,3)} - x_1^{(1,2)}$	$x_3^{(1,2,3)} - x_3^{(2,3)}$
1	cyclohexane	70	46.9	45.2	39.2	37.0	−1.7	−2.2
2	benzene	40	49.5	45.8	28.1	25.5	−3.7	−2.6
3	isopropyl alcohol	10	53.0	49.0	17.0	14.0	−4.0	−3.0
1	cyclohexane	70	46.9	51.5	19.5	19.4	+4.6	−0.1
2	benzene	40	49.5	50.9	12.6	11.6	+1.4	−1.0
3	n-propyl alcohol	10	53.0	53.0	6.5	6.0	0.0	−0.5

or n-propanol (3) are given. In the next columns are given: the equilibrium temperatures; the values $x_1^{(1,2)}$, $x_1^{(1,2,3)}$, $x_3^{(2,3)}$, $x_3^{(1,2,3)}$; and the respective differences $x_1^{(1,2,3)} - x_1^{(1,2)}$, and $x_3^{(1,2,3)} - x_3^{(2,3)}$. According to our relations, these differences should be equal to zero. The deviations are not large.

I wish to express my thanks to Professor W. Świętosławski for his help and assistance.

<div align="center">**Summary**</div>

1. The composition of ternary homoazeotropes has been determined on the basis of the relations, governing the activity coefficients of regular solutions and by employing the relations previously derived, between the constants A_{12}, A_{23} and A_{13} of binary regular solutions.

2. According to the relations obtained, the mole fractions of components 1 and 3 in the ternary azeotrope (1, 2, 3) are the same as those corresponding to the respective binary azeotropes (1, 2) and (2, 3):

$$x_1^{(1,2,3)} = x_1^{(1,2)}; \qquad x_3^{(1,2,3)} = x_3^{(2,3)}.$$

Component 2 is characterised by the intermediate internal pressure. The latter plays the role of the main azeotropic agent in Świętosławski's schema of formation of positive azeotropes [2], [3].

The compositions calculated are in good agreement with those determined experimentally [8] in systems formed by benzene, cyclohexane, isopropanol and n-propanol.

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Boiling Temperatures of Positive Ternary Azeotropes

by

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1. Introductory remarks. A set of basic equations was derived previously [1], determining the azeotropic point in the case of regular solutions:

$$(1a) \qquad \Delta S_1(T_{Az}^{(1,2,3)} - T_1) + A_{12}(1-x_1)^2 + A_{23}x_3^2 = 0,$$

$$(1b) \qquad \Delta S_2(T_{Az}^{(1,2,3)} - T_2) + A_{12}x_1^2 + A_{23}x_3^2 = 0, \qquad P = \text{const.}$$

$$(1c) \qquad \Delta S_3(T_{Az}^{(1,2,3)} - T_3) + A_{12}x_1^2 + A_{23}(1-x_3)^2 = 0.$$

The symbols represent: ΔS_1, ΔS_2, ΔS_3 — the vaporisation entropies of pure components at the respective boiling temperatures T_1, T_2, T_3 under a deliberately chosen pressure P; x_1, x_2, x_3 — the mole fractions of the components $1, 2, 3$ of the azeotropic mixture, boiling at $T_{Az}^{(1,2,3)}$; A_{12} and A_{23} — the regular solution constants of binary systems 1, 2 and 2, 3.

This simple set of equations was derived from the empirical relation between the mentioned constants [2]:

$$(2) \qquad\qquad A_{12} + A_{23} = A_{13}.$$

In the case of equal vaporisation entropies of the components, relation (2) is equivalent to the relation:

$$(3) \qquad\qquad Z_{12} + Z_{23} = Z_{13}.$$

From the above it is obvious that component 2 plays in the system a special role. It is characterised by an intermediate internal pressure compared with those of components 1 and 3. This may be seen from relation (2), since always $A_{12} < A_{13}$ and $A_{23} < A_{13}$, all the constants being positive. In recent papers [1], [2], component 2 has been identified with the so-called main azeotropic agent, a notion introduced by Świętosławski [3], [4] in his considerations of series of polyazeotropic systems.

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The purpose of the present paper is to calculate the boiling temperatures of ternary azeotropes on the basis of the properties of binary systems, formed by the components of the ternary system.

2. Boiling temperature of the ternary azeotrope. The considerations which follow are based on the previously derived [1] relations determining the composition of the ternary azeotrope:

$$(4) \qquad x_1^{(1,2,3)} = x_1^{(1,2)},$$

$$(5) \qquad x_3^{(1,2,3)} = x_3^{(2,3)},$$

where $x_1^{(1,2,3)}$ and $x_3^{(1,2,3)}$ are the mole fractions of components 1 and 3 in the ternary azeotrope $1, 2, 3$ and $x_1^{(1,2)}$, $x_3^{(2,3)}$ in the binary $1, 2$ and $2, 3$ ones.

The relations (4) and (5) are deduced from equations (1) with the assumption of the identity of vaporisation entropies of the components. Substituting relations (4) and (5) into (1b), we obtain

$$(6) \qquad \Delta S\left(T_{Az}^{(1,2,3)} - T_2\right) + A_{12}\left(x_1^{(1,2)}\right)^2 + A_{23}\left(x_3^{(2,3)}\right)^2 = 0,$$

where ΔS represents equal vaporisation entropy for all the components. From the above the following important relation results:

$$(7) \qquad T_{Az}^{(1,2,3)} = T_2 - \frac{A_{12}}{\Delta S} \cdot \left(x_1^{(1,2)}\right)^2 - \frac{A_{23}}{\Delta S} \cdot \left(x_3^{(2,3)}\right)^2.$$

We may transform this relation by introducing the value of the azeotropic ranges Z_{12} and Z_{23}, which are determined as follows [5], [6]:

$$(8) \qquad \frac{1}{2}\, Z_{12} = \frac{A_{12}}{\Delta S}; \qquad \frac{1}{2}\, Z_{23} = \frac{A_{23}}{\Delta S};$$

hence relation (7) may be expressed in the form:

$$(9) \qquad T_{Az}^{(1,2,3)} = T_2 - \tfrac{1}{2} Z_{12}\left(x_1^{(1,2)}\right)^2 - \tfrac{1}{2} Z_{23}\left(x_3^{(2,3)}\right)^2.$$

We may transform equation (9), replacing the azeotropic concentrations by the respective azeotropic depressions. Taking advantage of the following equations correlating these parameters [5], [6]:

$$(10) \qquad T_2 - T_{Az}^{(1,2)} = \tfrac{1}{2} Z_{12}\left(x_1^{(1,2)}\right)^2,$$

$$(11) \qquad T_2 - T_{Az}^{(2,3)} = \tfrac{1}{2} Z_{23}\left(x_3^{(2,3)}\right)^2,$$

in which $T_{Az}^{(1,2)}$ and $T_{Az}^{(2,3)}$ are the boiling temperatures of the azeotropes, and substituting them into relation (9), we obtain:

$$(12) \qquad T_{Az}^{(1,2,3)} = T_2 - \left(T_2 - T_{Az}^{(1,2)}\right) - \left(T_2 - T_{Az}^{(2,3)}\right).$$

Or, using the respective values for the azeotropic depressions, $\delta_{123} = T_2 - T_{Az}^{(1,2,3)}$ in the case of the ternary azeotrope, and $\delta_{12} = T_2 - T_{Az}^{(2,3)}$ for ~~binary azeotropes, we obtain~~ the following simple relation:

$$(13) \qquad\qquad \delta_{123} = \delta_{12} + \delta_{23}.$$

This relation may be formulated in the following way: "The azeotropic depression of the ternary azeotrope is equal to the sum of the azeotropic depressions of the two binary azeotropes formed with the main azeotropic agent, taking all these depressions with respect to this agent".

3. Experimental verification. In order to verify the usefulness of equation (13) we applied all the experimental data for the ternary azeotropes found in Horsley's Tables [7], except those containing water. As was to be expected, our assumptions were certainly not fulfilled in some cases, at least in the case of systems composed of highly polar substances. Highly polar molecules are capable of taking part in specific molecular interactions. Such solutions cannot be quantitatively described by any regular solution theory, including the semi-empirical one. The occurrence of specific molecular interactions may be also inferred from the deviations of experimental azeotropic ranges from (3). Hence in these cases neither relation (13), nor (2), nor (3) are fulfilled, provided that an unusual compensation of opposite deviations does not occur. Since we expect that relations (13) and (3) are simultaneously fulfilled, they are both submitted to experimental verification. In addition, on the basis of (3), it may be deduced, which component must be considered as the main azeotropic agent of the given system.

In Table I are presented the experimental data and the calculated values for 42 known ternary systems. In the column are given:

1) the number of components (2 — the main azeotropic agent),

2) the names of the components,

3) their boiling temperatures T_i,

4) the symbols characterising the binary systems: 1,2; 2,3; 1,3,

5) the boiling temperatures of the respective binary azeotropes T_{Az},

6) the boiling temperature differences Δ of the components forming the respective binary systems,

7) the azeotropic depression δ with respect to the lower boiling component,

8) the half-value of the symmetrical azeotropic range $\frac{1}{2}Z$, calculated according to the previously introduced relation [5], [6].

$$(14) \qquad\qquad \Delta = -\tfrac{1}{2}Z + \sqrt{2Z} \cdot \sqrt{\delta}.$$

The next columns contain the following data: 9) the half-values $\frac{1}{2}(Z_{12} + Z_{23})$, 10) differences $\Delta Z = \frac{1}{2}(Z_{13} - Z_{12} - Z_{23})$, which according to (3) should be equal to zero; 11) and 12) the boiling temperatures $T_{Az}^{(1,2,3)}$

TABLE 1

No.	No.	Components	T_i	Syst.	T_{Az}	Δ	δ	$\frac{1}{2}Z$	$\frac{1}{2}(Z_{12}+Z_{23})$	ΔZ	$T^{(1,2,3)}_{Az\ calc.}$	$T^{1,23}_{Az\ exp.}$	ΔT
1	1	Carbon disulphide	46.25	1.2	41.5	3.65	1.1	10.5	65.0	+1.0	36.7	35.95	+0.75
	2	Iodomethane	42.6	2.3	37.8	22.1	4.8	54.5					
	3	Methanol	64.7	1.3	37.65	18.45	8.6	66.0					
2	1	Carbon disulphide	46.25	1.2	41.5	3.65	1.1	10.5	22.5	+5.0	38.35	37.2	+1.15
	2	Iodomethane	42.6	2.3	39.45	0.35	2.8	12.0					
	3	Methylal	42.25	1.3	37.25	4.0	5.0	27.5					
3	1	Carbon disulphide	46.25	1.2	37.85	7.85	0.55	13.0	66.0	0.0	34.45	33.92	+0.53
	2	Bromoethane	38.4	2.3	35.0	26.3	3.4	53.0					
	3	Methanol	64.7	1.3	37.65	18.45	8.6	66.0					
4	1	Carbon disulphide	46.25	1.2	39.55	10.75	6.7	46.0	71.0	−5.0	36.55	37.0	−0.45
	2	Methyl acetate	57.0	2.3	54.0	7.7	3.0	25.0					
	3	Methanol	64.7	1.3	37.65	18.45	8.6	66.0					
5	1	Carbon disulphide	46.25	1.2	42.05	0.35	4.2	18.0	72.0	−6.0	36.05	37.0?	−0.95
	2	1-Chloropropane	46.6	2.3	40.6	18.1	6.0	54.0					
	3	Methanol	64.7	1.3	37.65	18.45	8.6	66.0					
6	1	Carbon disulphide	46.25	1.2	37.85	7.85	0.55	13.0	32.0	+21.5	29.30	24.7?	+4.6
	2	Bromoethane	38.4	2.3	29.85	6.5	2.05	19.0					
	3	Methyl formate	31.9	1.3	24.75	14.35	7.15	53.5					
7	1	Carbon disulphide	46.25	1.2	37.25	4.0	5.0	27.5	64.5	+1.5	36.82	35.55	+1.27
	2	Methylal	42.25	2.3	41.82	22.45	1.43	37.0					
	3	Methanol	64.7	1.3	37.65	18.45	8.6	66.0					
8	1	Carbon disulphide	46.25	1.2	36.5	9.1	0.65	15.5	56.0	−2.5	23.65	24.0?	−0.35
	2	2-Methyl-2-butene	37.15	2.3	24.3	5.25	7.6	40.5					
	3	Methyl formate	31.9	1.3	24.75	14.35	7.15	53.5					
9	1	Carbon disulphide	46.25	1.2	35.7	10.1	0.45	15.5	64.0	−10.5	21.35	21.5?	−0.15
	2	Pentane	36.15	2.3	21.8	4.25	10.1	48.5					
	3	Methyl formate	31.9	1.3	24.75	14.35	7.15	53.5					
10	1	Carbon disulphide	46.25	1.2	42.05	0.35	4.2	18.0	29.5	+12.5	41.7	38.2?	+3.5
	2	1-Chloropropane	46.6	2.3	46.25	7.5	0.35	11.5					
	3	Ethyl formate	54.1	1.3	39.35	7.85	6.9	42.0					

No.		Component												
11	1	Carbon disulphide	46.25	1.2	36.5	9.10	0.65	15.5	33.0	−5.5	34.55	35.2	−0.65
	2	2-Methyl-2-butene	37.5	2.3	35.20	4.75	2.3	17.5					
	3	Methylal	42.25	1.3	37.25	4.0	5.0	27.5					
12	1	Hexane	68.95	1.2	59.95	7.95	1.05	16.5	48.5	−0.5	58.3	58.3?	0.0
	2	Chloroform	61.0	2.3	59.35	17.3	1.65	32.0					
	3	Ethanol	78.3	1.3	58.68	9.35	10.27	48.0					
13	1	Iodomethane	42.6	1.2	39.45	0.35	2.8	12.0	49.8	+4.7	39.02	38.5	+0.52
	2	Methylal	42.25	2.3	41.82	22.45	0.43	29.5					
	3	Methanol	64.7	1.3	37.8	22.1	4.8	54.5					
14	1	Propyl acetate	101.55	1.2	100.75	0.65	0.8	4.5	15.0	0.0	97.65	99?	−1.35
	2	3-Pentanone	102.2	2.3	99.1	1.0	2.1	10.5					
	3	Nitromethane	101.2	1.3	97.6	0.35	3.6	15.0					
15	1	2-Methyl-2-butene	37.15	1.2	35.0	1.25	2.15	11.0	64.0	+1.0	31.6	31.4	+0.2
	2	Bromoethane	38.4	2.3	35.0	26.3	3.4	53.0					
	3	Methanol	64.7	1.3	31.75	27.55	5.4	65.0					
16	1	Methyl acetate	57.0	1.2	55.0	0.75	1.25	6.5	22.0	−0.5	54.25	53.9	+0.35
	2	Acetone	56.25	2.3	55.5	8.45	0.75	15.5					
	3	Methanol	64.7	1.3	54.0	7.7	2.25	21.5					
17	1	1-Chloro-2-methylpropane	68.85	1.2	53.05	12.0	3.2	32.5	47.5	−3.5	52.3	52.0	+0.3
	2	Acetone	56.25	2.3	55.5	8.45	0.75	15.5					
	3	Methanol	64.7	1.3	55.75	4.15	8.95	44.0					
18	1	Cyclohexane	80.75	1.2	53.0	24.5	3.25	50.0	65.5	+5.5	52.25	51.1	+1.15
	2	Acetone	56.25	2.3	55.5	8.45	0.75	15.5					
	3	Methanol	64.7	1.3	54.0	16.05	10.7	71.0					
19	1	Isoprene	34.1	1.2	32.0	4.3	2.1	16.0	35.0	+15.5	23.45	<23	>0.45
	2	Bromoethane	38.4	2.3	29.85	6.5	2.05	19.0					
	3	Methyl formate	31.9	1.3	22.5	2.2	11.6	50.5					
20	1	Tetrachloroethylene	120.8	1.2	118.55	5.2	2.25	18.0	28.5	+7.5	117.85	<116.0	>1.85
	2	Ethyl carbonate	126.0	2.3	125.3	5.0	0.7	10.5					
	3	Isoamyl alcohol	131.0	1.3	116.2	10.2	4.6	36.0					
21	1	Tetrachloroethylene	120.8	1.2	118.75	3.2	2.05	14.0	22.5	−5.5	117.75	117.6	+0.15
	2	Paraldehyde	124.0	2.3	123.0	0.4	0.6	8.5					
	3	Isoamyl formate	123.6	1.3	117.9	2.8	2.9	17.0					

No.	No.	Components	T_i	Syst.	T_{Az}	Δ	δ	$\frac{1}{2}Z$	$\frac{1}{2}(Z_{12}+Z_{23})$	ΔZ	$T^{(1,2,3)}_{Az\,calc.}$	$T^{1,23}_{Az\,exp.}$	ΔT
22	1	1,2-dibromoethane	131.0	1.2	130.05	0.8	0.95	15.5	37.5	+1.5	127.15	127.5	−0.35
	2	Chlorobenzene	131.8	2.3	128.9	8.9	1.9	22.0					
	3	Propionic acid	140.7	1.3	127.75	9.7	3.25	39.0					
23	1	2-Methyl-2-butene	37.15	1.2	35.0	1.25	2.15	11.0	30.0	+10.5	26.45	24.1	+2.35
	2	Bromoethane	38.4	2.3	29.85	6.5	2.05	19.0					
	3	Methyl formate	31.9	1.3	24.3	5.25	7.6	40.5					
24	1	2-Methyl-butane	27.95	1.2	23.7	10.45	4.25	35.0	54.0	−2.5	15.15	16.95	−1.8
	2	Bromoethane	38.4	2.3	29.85	6.5	2.05	19.0					
	3	Methyl formate	31.9	1.3	17.05	3.95	10.9	51.5					
25	1	Pentane	36.15	1.2	<32.8	2.25	>3.35	>17.5	>36.5	<+12.5	<24.25	21.7	<+2.55
	2	Bromoethane	38.4	2.3	29.85	6.5	2.0	19.0					
	3	Methyl formate	31.9	1.3	21.8	4.45	10.1	49.0					
26	1	2-Methyl-2-butene	37.15	1.2	32.95	1.35	2.85	14.0	41.0	−0.5	24.15	24	+0.15
	2	Ethanethiol	35.8	2.3	27.0	3.9	4.9	27.0					
	3	Methyl formate	31.9	1.3	24.3	5.25	7.6	40.5					
27	1	2-Methyl-2-butene	37.15	1.2	34.2	2.55	0.4	5.5	25.5	+15.0	27.8	24	+3.8
	2	Ethyl ether	34.6	2.3	28.2	2.7	3.7	20.0					
	3	Methyl formate	31.9	1.3	24.3	5.25	7.6	40.5					
28	1	Pentane	36.15	1.2	33.4	1.55	1.2	7.5	27.5	+21.5	27.0	20.4	+6.6
	2	Ethyl ether	34.6	2.3	28.2	2.7	3.7	20.0					
	3	Methyl formate	31.9	1.3	21.8	4.25	10.1	49.0					
29	1	Cyclohexane	80.75	1.2	72.8	3.7	4.25	24.0	47.5	+11.0	67.55	64.3	+3.25
	2	Ethyl acetate	77.05	2.3	71.80	1.25	5.25	23.5					
	3	Ethyl alcohol	78.3	1.3	64.9	2.45	13.4	58.5					
30	1	d-Limonene	177.8	1.2	171.35	6.55	6.45	38.0	74.5	+13.5	167.55	162.45	+5.1
	2	Aniline	184.35	2.3	180.55	13.05	3.8	36.5					
	3	Glycol	197.4	1.3	163.5	19.6	14.3	90.0					
31	1	d-Limonene	177.8	1.2	166.35	6.05	5.4	32.5	44.5	−3.5	164.6	165.5	−0.9
	2	Propyl lactate	171.75	2.3	170.0	2.75	1.75	12.0					
	3	1,3-Dichloro-2-propanol	174.5	1.3	165.75	3.3	8.75	41.0					

No.		Component											
32	1	Cyclohexane	80.75	1.2	72.8	3.7	4.25	24.0					
	2	Ethyl acetate	77.05	2.3	74.0	5.4	3.05	21.5	45.5	+6.5	69.75	68.3	+1.45
	3.	Isopropyl alcohol	82.45	1.3	68.6	1.7	12.15	52.0					
33	1	Cyclohexane	80.75	1.2	77.4	0.55	2.8	12.0					
	2	Benzene	80.2	2.3	77.12	17.0	3.08	39.0	51.0	+3.0	74.32	<74	> +0.32
	3	Propyl alcohol	97.2	1.3	74.3	16.45	6.45	54.0					
34	1	Mesitylene	164.0	1.2	156.3	3.35	4.35	23.5					
	2	Cyclohexanol	160.65	2.3	155.6	2.65	5.05	25.0	48.5	−13.0	151.25	<154.5	> −3.25
	3	Methyl oxalate	163.3	1.3	154.8	0.7	8.5	35.5					
35	1	a-pinene	155.8	1.2	153.4	0.3	2.4	10.0					
	2	Bromobenzene	156.1	2.3	155.3	2.1	0.8	7.0	17.0	+1.0	152.6	<152.3	> +0.3
	3	Ethyl bromoacetate	158.2	1.3	152.5	2.4	3.3	18.0					
36	1	a-Pinene	155.8	1.2	150.45	1.95	3.4	17.0					
	2	Anisole.	153.85	2.3	153.8	4.35	0.05	5.0	22.0	−4.0	150.4	<150.4	> 0.0
	3	Ethyl bromoacetate	158.2	1.3	152.5	2.4	3.3	18.0					
37	1	a-Pinene	155.8	1.2	153.4	0.3	2.4	10.0					
	2	Bromobenzene	156.1	2.3	148.6	1.75	5.75	26.5	36.5	−3.0	145.9	146.4	−0.5
	3	Isobutyric acid	154.35	1.3	146.7	1.45	7.65	33.5					
38	1	a-Pinene .	155.8	1.2	150.45	1.95	3.4	17.0					
	2	Anisole	153.85	2.3	149.0	0.5	4.85	21.0	38.0	−4.5	145.6	143.9	+1.7
	3	Isobutyric acid	154.35	1.3	146.7	1.45	7.65	33.5					
39	1	d-Limonene	177.8	1.2	171.2	1.4	6.6	29.0					
	2	Benzaldehyde	179.2	2.3	174.5	2.7	2.0	13.0	42.0	−9.0	166.5	168.7	−2.2
	3	Isovaleric acid	176.5	1.3	168.9	1.3	7.6	33.0					
40	1	d-Limonene	177.8	1.2	174.8	1.55	3.00	15.0					
	2	a-Chlorotoluene	179.35	2.3	171.2	2.85	5.3	26.5	41.5	−8.5	166.55	168.7	−2.15
	3	Isovaleric acid	176.5	1.3	168.9	1.3	7.6	33.0					
41	1	Diisoamyloxy methane	207.5	1.2	206.5	3.35	1.0	9.5					
	2	Nitrobenzene	210.85	2.3	204.2	5.35	1.3	14.0	23.5	+7.5	199.85	197?	+2.85
	3	Benzyl alcohol	205.5	1.3	198.7	2.0	6.8	31.0					
42	1	d-Limonene	177.8	1.2	174.8	1.55	3.0	15.0					
	2	a-Chlorotoluene	179.35	2.3	178.0	2.80	1.35	10.0	25.0	+4.0	173.45	172.5	+0.95
	3	Isobutyl acetate	182.15	1.3	172.5	4.35	5.3	29.0					

of the ternary azeotropes, calculated from relation (13), and that found in Horsley's Tables. In the last column the differences ΔT between these values are given.

On the basis of the data presented in Table I, it is possible to distinguish three characteristic groups.

In the first group 21 systems out of 42 (1, 3, 4, 5, 8, 9, 11, 12, 13, 15, 16, 17, 21, 22, 26, 31, 33, 35, 37, 42) show a satisfactory agreement between the calculated and experimentally found values of $T_{Az}^{(1,2,3)}$. The deviations lie within the limit of $\pm 1°C$. For the same group, values of ΔZ show deviations of the order of $\pm 6°C$.

The systems belonging to the second group (6, 10, 23, 25, 27, 28, 29, 30, 41) show greater deviations from both relations (13) and (3). The calculated values of $T_{Az}^{(1,2,3)}$ and Z_{13} are always greater than $(Z_{12} + Z_{23})$ (ΔT and ΔZ are both positive). It seems that the sign of these deviations indicates the existence of strong specific interactions between unlike molecules of the systems 1,2 or 2,3.

The deviations observed in systems of the third group (34, 29, 40) are interesting. Values of ΔT and ΔZ are both negative. However, for a more detailed analysis of such deviations at least the differences in the vaporisation entropies should be taken into consideration.

The remaining 9 systems (2, 7, 14, 18, 19, 20, 24, 32, 38) belong to a type which does not possess a suitable background for a reasonable classification.

One remark has to be added. The ternary azeotropes examined up to the present do not represent any systematically collected material, characterised by adequately chosen components. If there were data available for series of systems such as (A, B, H_i) (H_i — being representative of a homologous series), more conclusions might have been made.

I wish to express my thanks to Professor W. Świętosławski for a helpful discussion.

Summary

1. On the basis of the properties of regular solutions and with the aid of the equations previously derived [1], [2], a new one enabling to calculate the boiling temperature of the ternary azeotrope $(T_{Az}^{(1,2,3)})$ has been derived. This temperature has been correlated with the composition of the binary azeotropes and their boiling temperatures under the same pressure $(T_{Az}^{(1,2)})$ and $(T_{Az}^{(2,3)})$. The relation between the mentioned boiling temperatures has the form:

(13)
$$\delta_{123} = \delta_{12} + \delta_{23},$$
where
$$\delta_{123} = T_2 - T_{Az}^{(1,2,3)},$$
$$\delta_{12} = T_2 - T_{Az}^{(1,2)},$$
$$\delta_{23} = T_2 - T_{Az}^{(2,3)},$$

T_2 being the boiling temperature of the main azeotropic agent [3], [4], characterised by a medium value of the internal pressure.

2. The relations have been tested for 42 ternary systems given in Horsley's Tables. For 21 systems the calculated values of $T_{Az}^{(1,2,3)}$ agree within $\pm 1^{\circ}C$. Greater deviations found in the remaining systems are partly due to specific interactions between unlike molecules, and partly to other factors, for instance, to the differences in vaporisation entropies.

RAW ORGANIC MATERIALS DEPARTMENT OF THE INSTITUTE OF PHYSICAL CHEMISTRY, POLISH ACADEMY OF SCIENCES

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On Hydrogen Bonds in Some Nitroalcohols. II.
Infra-Red Absorption Bands of Nitrogroups

by

T. URBAŃSKI

Communicated at the meeting of March 19, 1956

In the first paper on infra-red absorption spectra of some nitro-alcohols [1] the former hypothesis on hydrogen bonds between the nitro-group and hydroxyl groups [2] was subjected to a more detailed exa-
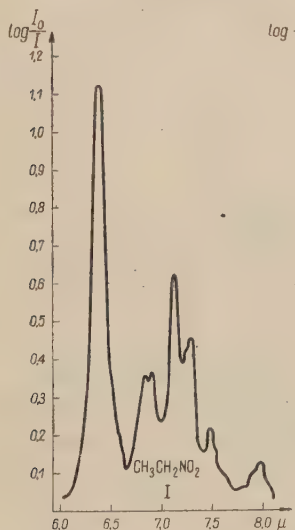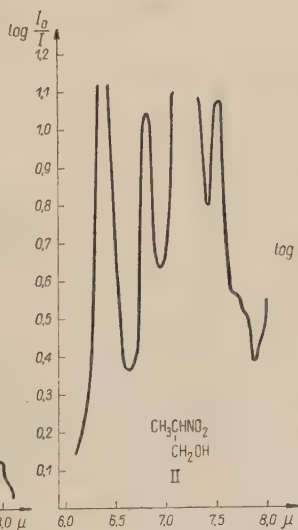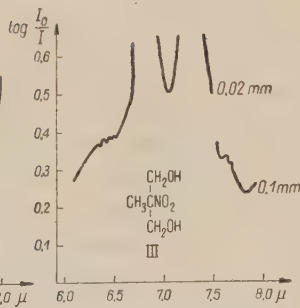
Fig. 1   Fig. 2   Fig. 3

mination, mainly on the basis of the frequency and intensity of absorp-tion bands produced by the hydroxyl groups.

In the present paper, the n_8 absorption band (see [1]), near $6.4\,\mu$, is examined in a more detailed manner, and the n_8' band, near $7.5\,\mu$ (not described in the first paper) is also subjected to consideration.

Figures 1-9 give the absorption curves for the range between c. 6 and $8\,\mu$.

In the compounds (I) (nitroethane), (II) (2-nitropropanol), (IV) (1-ni-tropropane), (V) (2-nitrobutanol) and (VII) (2-nitropropane), i. e. nitro-paraffins or monohydroxylic alcohols with active hydrogen atom atta-
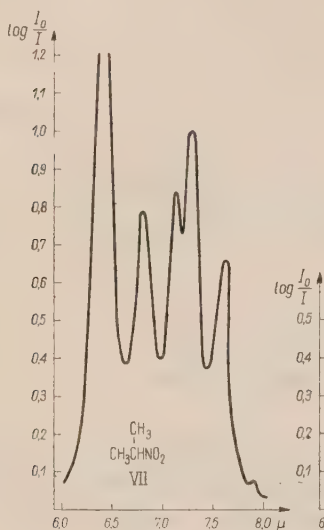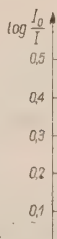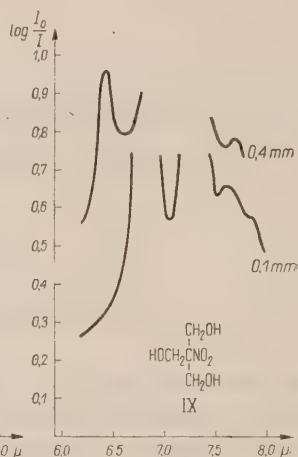
Fig. 4   Fig. 5   Fig. 6
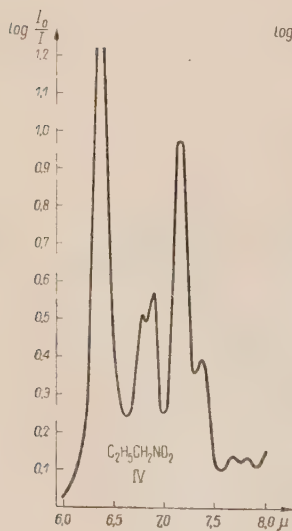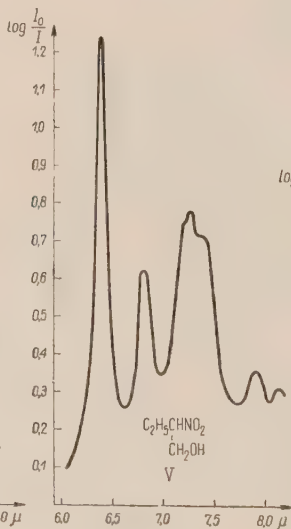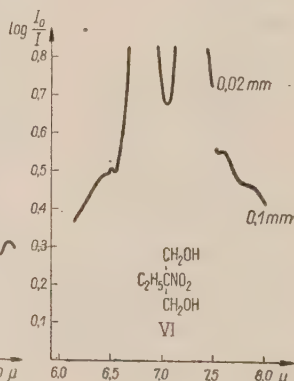
Fig. 7   Fig. 8   Fig. 9

ched to the carbon atom bonded with the nitrogroup, the frequency is 1567 cm.$^{-1}$. In the alcohols with two hydroxyl groups, i. e. (III) (2-nitro-2-methylpropane-1,3-diol) and (VI) (2-nitro-2-ethylpropane-1,3-diol) or in the alcohol with one hydroxyl group and no active hydrogen attached to the carbon atom bonded with the nitrogroup, i. e. (VIII) (2-nitro-2-methylpropanol), the intensity of the n_8 band is very small, the band

becomes broader and the frequency lower: 1543, 1534 and 1555 cm.$^{-1}$ respectively (see Table I).

TABLE I (n_3 Band)

Suspension in paraffin-oil

Number	Formula	Thickness of the layer mm.	n_8 Band		
			Wave length μ	Frequency cm^{-1}.	$\log \dfrac{I_0}{I}$
(I)	$CH_3CH_2NO_2$	0.02	6.38	1567	1.131
(II)	$CH_3CH\ NO_2$ $\mid$ CH_2OH	0.02	6.38	1567	1.420
(III)	CH_2OH $\mid$ $CH\ C\ NO_2$ $\mid$ CH_2OH	0.02	6.48	1543	0.383
(IV)	$C_2H_5CH_2NO_2$	0.02	6.38	1567	1.337
(V)	$C_2H_5CH\ NO_2$ $\mid$ CH_2OH	0.02	6.38	1567	1.244
(VI)	CH_2OH $\mid$ $C_2H_5C\ NO_2$ $\mid$ CH_2OH	0.02	6.52	1534	0.494
(VII)	CH_3 $\mid$ $CH_3CH\ NO_2$	0.02	6.38	1567	1.444
(VIII)	CH_3 $\mid$ $CH_3C\ NO_2$ $\mid$ CH_2OH	0.02	6.43	1555	0.174
(IX)	CH_2OH $\mid$ $HOCH_2\ C\ NO_2$ $\mid$ CH_2OH	0.1 0.4	6.43 ,,	1555 ,,	0.350 0.975

The absorption spectrum of the triol (IX) (2-nitro-2-hydroxymethyl-propane-1,3-diol) gives a striking example of a very weak NO_2 band. Only when a considerable thickness of layer of the sample (0.4 mm.) was applied, did the presence of the n_8 band become clear.

The presence of one hydroxyl group reduced slightly the frequency of the n'_8 band in all examined compounds. Thus, by passing from nitro-ethane (I) to the corresponding monohydroxylic alcohol (II), $\Delta\nu$ was 9 cm.$^{-1}$. In the instance of 1-nitropropane (IV) or 2-nitropropane (VII) and the alcohols (V) or (VIII) respectively, the decrease of the frequency $\Delta\nu$ was 6 and 5 cm.$^{-1}$ respectively. There was no decrease of the intensity of the absorption, on the contrary -- an increase of the intensity of the absorption was noticed.

The presence of two hydroxyl groups markedly reduced the frequency of the band. Thus, the difference between the n'_8 frequencies in

TABLE II (n_8' Band)

Suspension in paraffin-oil

Number	Thickness of the layer mm.	n_8' Band		
		Wave length μ	Frequency cm.$^{-1}$	$\log \frac{I_0}{I}$
(I)	0.02	7.45	1342	0.208
(II)	0.02	7.50	1333	1.066
(III)	0.1	7.63	1310	0.329
(IV)	0.02	7.35	1361	0.396
(V)	0.02	7.38	1355	0.719
(VI)	0.1	7.58	1319	0.554
(VII)	0.02	7.60	1316	0.664
(VIII)	0.02	7.63	1311	0.294
(IX)	0.1	7.58	1319	0.654
	0.4	7.63	1311	0.783

compounds (I) and (III), (IV) and (VI) was 32 and 42 cm.$^{-1}$ respectively. At the same time, intensity of the band considerably decreased and, for a better measurement, a thicker layer of the suspension had to be applied.

The same low frequency of the n_8' band (1311 - 1319 cm.$^{-1}$) was noticed with compound (IX) containing three hydroxylic groups, the intensity of the absorption being also relatively small.

Discussion

It was formerly noticed that the maximum of ultraviolet absorption near 270 $m\mu$ due to NO_2 group was transformed into a more or less pronounced band under the influence of two hydroxyl groups. This was explained by the formation of six-membered chelate rings, containing hydrogen bonds between OH groups and the NO_2 group [2], according to diagramm (C). Further support to this hypothesis was given by the examination of the frequencies and intensities of OH bands in the infrared absorption spectrum [1], and the experimental facts described in the present paper seem to confirm this view.

Here are the main points in favour of the hypothesis on hydrogen bonds between OH and NO_2 groups (in addition to those mentioned in the first publication [1]):

1. The frequency and intensity of the n_8 and n_8' bands are considerably reduced in presence of at least two hydroxyl groups.

2. One hydroxyl group slightly reduces the frequency of the n_8' band and in the compounds (II) and (V) no decrease of the intensity of the bands has been noticed.

3. One hydroxyl group without an active hydrogen atom attached to the carbon bonded with the nitrogroup (compound VIII), reduces the frequency and intensity of the n_8 band. The intensity of the n_8' band is also considerably reduced.

It remains unexplained why in compound (VIII) one hydroxyl group produced such a strong effect upon the intensity of the n_8 band, whereas in the compounds (II) and (V) no lowering of the intensity of the n_8 band was noticed. The only explanation we can give for the time being, is based on the difference between the structure of the compound (VIII) on one side and the compounds (II) and (V) on the other. The former does not contain, while the two latter compounds contain the active hydrogen atom attached to the carbon atom bearing the nitrogroup. The active hydrogen either weakens the hydrogen bonds of type (C) (as it was admitted in the previous paper [1]), or increases the intensity of NO_2-vibrations.

It seems also that the substances in solid form possess another kind of hydrogen bonds between the OH and the NO_2 groups, in addition to the bonds according to (C). This would be the intermolecular bonds of the type (F).

(C)   (F)
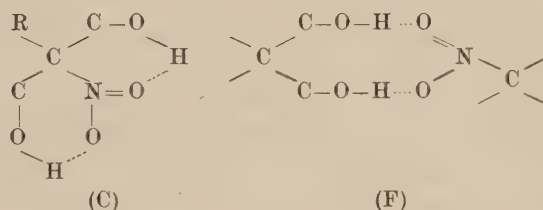
The (F) bonds would increase the effect of the (C) bonds upon the infra-red absorption spectrum. In order to establish the correctness of diagram (F), experiments will be carried out on the infra-red spectra of the compounds in question in solution.

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The Influence of the Thickness of the Adsorption Layer on the Potential of the Microelement Used in Chromatography

by

B. KAMIEŃSKI and I. KULAWIK

Presented by B. KAMIEŃSKI on March 26, 1956

The minute indicator electrode described in former papers [1]-[13] is so sensitive in its applications to chromatography and to the detection of atmospheric contaminations that it appears useful to publish some further details on the optimal conditions of its preparation. The thin adsorption layer of the microelement plays the role of an electrolyte and its resistance might be so great that even an electrometer of very high inner resistance (10^{16} Ω) would exhibit only a part of the potential of the electrical element. In the case of thick layers the diffusion of the active substances is more difficult through the layer to the cross-section of the wire (electrode) and the inertia of the element becomes greater. The potential of the electrode depends on the thickness of the layer and the sensitivity of the electrode is affected by the nature of the layer.

Thus the influence of thickness was investigated. The electrode of the microelement [1]-[8] was covered with titanic acid gel [7], [12]. For instance, a solution of titanic acid in concentrated hydrochloric acid containing 5.65 mg. Ti in 1 ccm. (or 9.43 mg. TiO_2 in 1 ccm.) was divided into three parts the concentration of which decreased as in geometrical progression (1:4:16). A volume of 0.25 ccm. of the titanic chloride was diluted in 50 ccm. of distilled water and the diluted solution divided into three equal parts, of 16 ccm. The first was dialysed immediately, the second was diluted four times and dialysed, the third was diluted 16 times and dialysed, each separately. The dialysis lasted 15 days and the water was changed twice daily. No anions of Cl′ could be detected in the dialysate after 15 days. However, the volume of the dialysates changed so that each was made up to 21 ccm. with water (the dialysate of the greatest volume). The titanic acid sols served for the preparation

of titanic acid gel layers of different thickness on an antimony micro-electrode and the potential of the electrode was measured against an antimony reference electrode. Different thicknesses of the layers were prepared by using different concentrations of titanic acid sols. One drop of the sol was placed on the top of the electrode and dried at room temperature. Thicker layers were prepared by using stronger concentrations

4cm

6 cm

CH₃COOH
or (NH₄)₂CO₃

Fig. 1

ot the sol, or eventually, by putting a second or several drops after the former dried up.
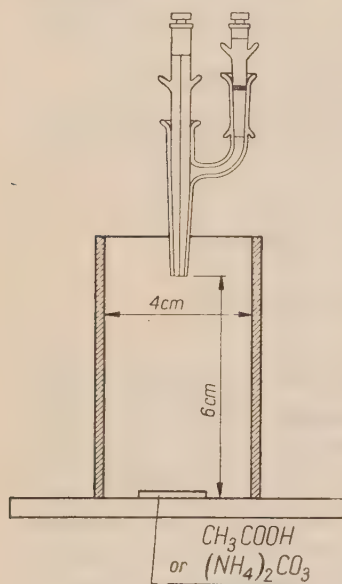
Fig. 1 represents the arrangement of the microelement. The distance of the electrode from the reagent is visible. As an adopted rule for checking the element, vapours of acetic acid and vapours of ammonium carbonate were used alternatively. A strip of filter paper (2 cm. × 2 cm.) was soaked with acetic acid and placed at the bottom as may be seen in Fig. 1. When the potential of the element was stabilised, 1 gram of ammonium carbonate was introduced after having replaced the strip of filter paper and after having cleaned the walls and bottom of the receptacle.

The thickness of the titanic acid gel layer may be computed approximately when taking into account the concentration of the used sol, its volume and the density of the gel. One drop of the sol contained 0.968×10^{-7} g. titanic oxide. Assuming a fairly high density of titanic oxide (3.8) as the extreme of the density of the gel, and taking into account the area of the dried gel, we obtain as a result the thickness of the layer which is at least 2.98×10^{-7} cm. However, notwithstanding its thinness, the layer exhibited a sufficient conductivity, as shown in Fig. 2.

The ordinate represents the potential of the electrode, the abscissa — the time of observation when the acid vapours of acetic acid, and later the basic vapours of ammonium carbonate, reached the microelectrode covered with titanic acid gel.

A thicker layer was prepared by drying subsequently 2 drops on the electrode. The sudden change of the potential is represented in Fig. 3.

No great changes were observed when the layer became thicker as a result of applying four drops of the sol and drying each before the next was put on the top of the electrode. The observations of the potential with time are represented in Fig. 4.

It should be underlined that the best results were obtained when two drops of sol were used to build the adsorption layer of titanic acid gel.

Thinner layers were prepared by applying titanic acid sol diluted four times. One drop of this solution gave a layer of great resistance
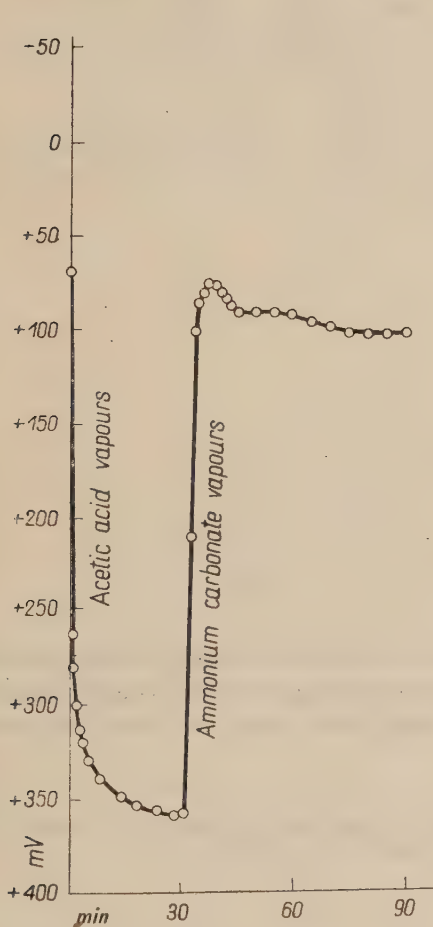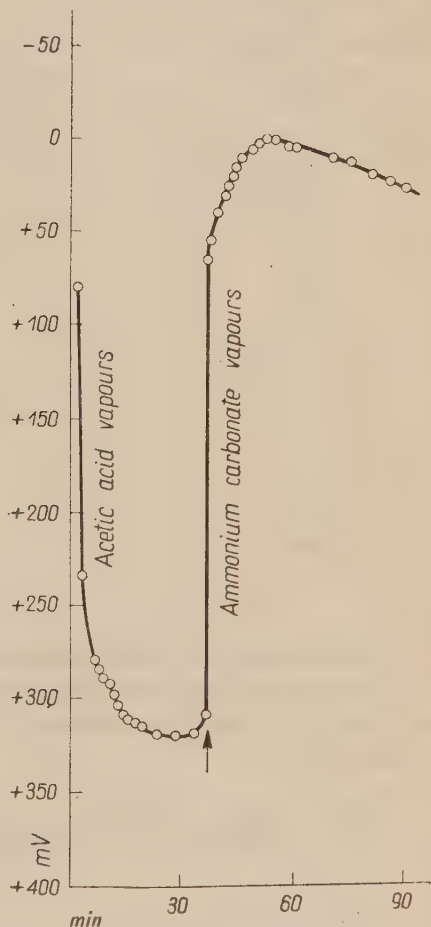

Fig. 2   Fig. 3

and the change of the potential was more rapid but much smaller owing to the great internal resistance of the element. The question whether more diluted solutions of titanic acid give rise to a smaller potential change seems quite clear, and out of a series of measurements only one is quoted here. When applying one drop of the sol diluted 16 times and drying it on the top of the electrode we obtained a layer which nevertheless gave potential changes as may be seen from Fig. 5; however, it is not advisable to use such thin layers.
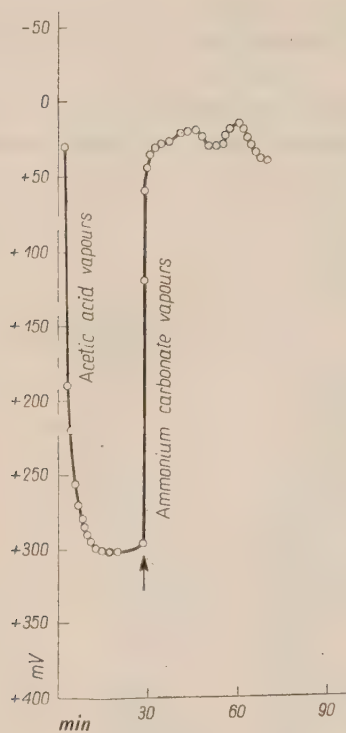
Fig. 4
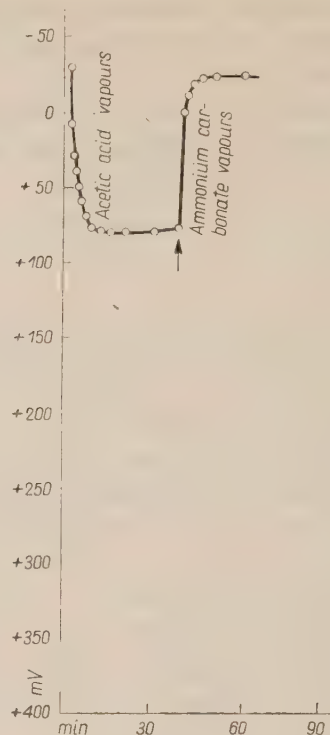
Fig. 5

Reassuming, we may conclude that there are optimal conditions of layer thickness, and that there is a certain tolerance of thickness so that it is easy to prepare a suitable layer using two drops of the strongest titanic acid sol.

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Observations on the Sedimentation of the Ciężkowice Sandstone near Ciężkowice (Carpathian Flysch)

by

L. KOSZARSKI

Presented by M. KSIĄŻKIEWICZ on March 19, 1956

Each stratigraphic member of the Carpathian Flysch may be characterised from the sedimentological point of view by an assemblage of structural and textural features, the observation of which may make possible a reconstruction of the conditions of deposition. The present paper concerns the sedimentary characteristics of one of the members of the Carpathian Flysch; it should, however, be made clear that the observations have been collected from a comparatively restricted area.

The Ciężkowice sandstone (upper part of the Paleocene - Lower Eocene) of the Silesian nappe consists of thick-bedded sandstones alternating with conglomerates, muddy gravels and shaly sandstones. Variegated, black or green shales with thin-bedded sandstones may also occur as intercalations. The series, about 250 m. thick, rests on the variegated shales and Upper Istebna beds of the Paleocene, and is covered by the Hieroglyphic beds and variegated shales of the upper part of the Eocene.

The characteristics of the series are based mainly on the analysis of bedding. Graded bedding is very common, but irregular bedding and lensing are equally frequent as well as current bedding, lamination and composite bedding, while slip bedding, wavy bedding and cross bedding are generally absent.

Graded bedding is represented by a few types, most of which correspond to the types from the Carpathian Flysch described by M. Książkiewicz [3]. Single normal bedding and single pensymmetrical bedding are frequently encountered; so are a few varieties of multiple graded bedding. Single inverted bedding and single symmetric inverted bedding are rare. In the multiple graded bedding, the following varieties may be distinguished: normal, pensymmetric, inverted and mixed type. The first three types are characterised by repeated occurrence in the bed of one of the varieties of single graded bedding, and the last type is a combi-

nation of two or more of various types of single grading. The separation
of grades in the Ciężkowice sandstone is in general very poor — on the
whole worse than in the underlying Istebna beds, and therefore it is
often difficult to distinguish graded from ungraded beds. Graded bed-
ding often combines with other types of bedding. The origin of grading
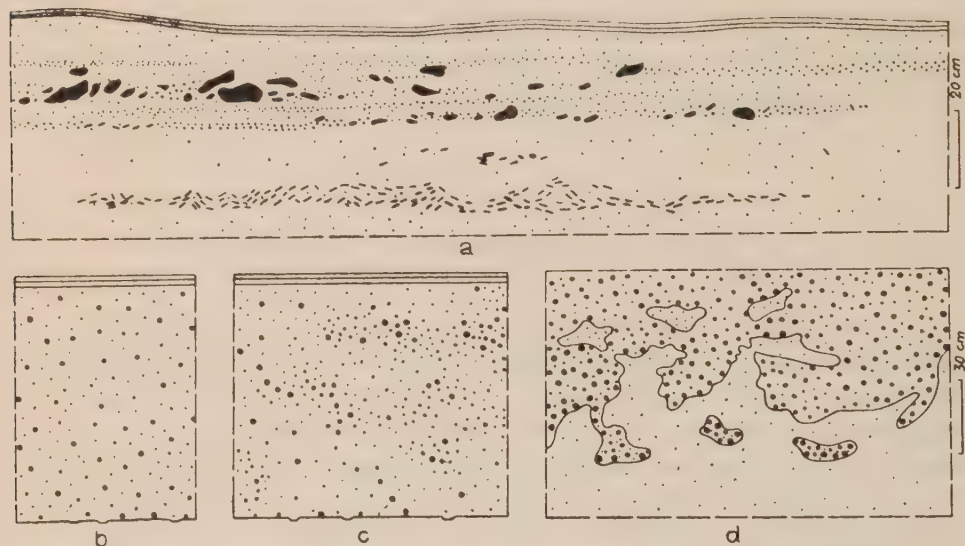is now regarded by many authors, after Ph. Kuenen and C. I. Miglio-

Fig. 1. Lenticular and ungraded bedding
a — Lenticular bedding (fragments of shales in black)
b — Regular ungraded bedding
c — Irregular ungraded bedding
d — Distortion of original bedding by mass movement

rini [1], as the result of turbidity currents, and various types of
grading are referred by M. Książkiewicz [3] to changes in velocity.
Various modifications of multiple grading may similarly be explained.
        Lenticular bedding is a very common feature in the Ciężkowice
sandstone, but it occurs only in thicker beds. In some cases it occurs
independently; more often, however, it combines with graded bedding.
Elongated strips and thin lenses of coarse grain within finer grains form
this kind of grading (Fig. 1a). Lenses are usually asymmetrical in longi-
tudinal section; grains are not regularly distributed either vertically or
horizontally; beds with lensing contain abundant fragments and balls
of shales (Fig. 1a). This type of bedding should, since it is always con-
nected with graded and ungraded beds, also be ascribed to turbidity
currents. It seems probable that some lenses derive from erosion and
dissipation of poorly consolidated sandy packets torn from the bottom
and dragged by currents, while others are an erosive feature.

Ungraded beds [3] are often found in the Ciężkowice sandstone; two varieties of this type of bedding probably of diverse origin may be distinguished:

a) regularly ungraded beds (Fig. 1b) with uniform distribution of all grades in the vertical section; sorting may be good (homogenous sandstones of S. Dżułyński and A. Radomski [5]) or poor; common features with graded beds point to a common cause, i. e. turbidity currents;

b) irregular ungraded beds with irregularly distributed accumulations of coarser material within the finer graded or vice versa (Fig. 1c). This type of bedding is formed possibly by distortions of graded beds or beds with lensing, e. g. by slumping of poorly consolidated fresh sand on the bottom. This type of bedding is observable only in thick beds.

Laminated bedding usually occurs combined with graded or ungraded beds, thus forming composite bedding. As a rule, lamination is parallel, and never wavy. Both types distinguished by M. Książkiewicz [3], normal and symmetrical, may be represented; a type of lamination is also present which can be denoted as multiple laminated bedding if thicker and finer laminae alternate twice or three times in the bed. Lamination is ascribed [3] to feeble secondary turbidity currents.

Fig. 2. Current bedding: a — oblique, b — parallel

Current bedding is very common in the Ciężkowice sandstone. Two types of current bedding may be distinguished, on the basis of the relation of diagonal laminae to the erosional surface on which they repose. Tentatively, they are called "oblique current bedding" and "parallel current bedding" respectively. In the first instance, the laminae are oblique to the lower erosional surface which is usually parallel to the bedding (Fig. 2a). In the second case, the laminae are parallel to the erosional surfaces which run slantwise in relation to the bedding (Fig. 2b). The first type, already described by M. Książkiewicz [2], [3], occurs very seldom and is connected with fine-grained and thin-bedded layers. On the other hand, the second type occurs frequently in the Ciężkowice sandstone, in its coarse-grained and conglomeratic portions. Both types were formed under the action of tractive bottom currents but there must have been a difference in the mechanism of deposition.

Composite bedding, i. e. a combination of graded and laminated bedding is frequent, represented by two varieties — normal and symme-
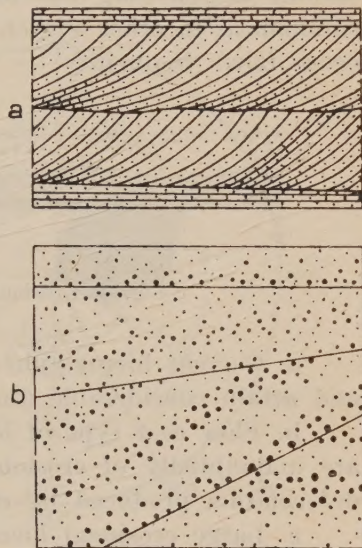
trical, of which the latter is less common. Frequent also are combinations of other types of bedding, as grading combined with lensing, lamination and current bedding, or ungraded mixed with lensing and lamination. It seems that the wealth of various types of bedding distinguishes the Ciężkowice sandstone from other Flysch series.

Hieroglyphs, especially of mechanical origin, furnish important indications determining the condition of sedimentation. They are not so common as in other Flysch series, but certain forms, enumerated below, occur fairly frequently:
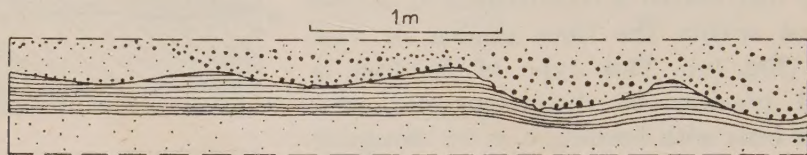

Fig. 3. Erosional hieroglyphs on the lower surface

a. Current hieroglyphs (= flow marks) are rare but well developed and attain considerable dimensions and variety of forms.

b. Flow cast type of hieroglyphs are fairly frequent. Some of them are undoubtedly of erosional origin but their ultimate form is due to deformation produced by compaction of the overlying bed.

c. Large erosional hieroglyphs (Fig. 3) form considerable ridges on the lower surfaces, up to 1.5 m. wide and some scores of cm. thick. They are connected with very thick beds only, reposing irregularly on the subjacent shales or shaly sandstones. The erosional hieroglyphs present some intermediate forms between the flow marks and wash-outs. They indicate a strong erosion of the bottom during the deposition of the Ciężkowice sandstone.

d. Biohieroglyphs are rare and feebly developed.

The composition of pebbles occurring in the Ciężkowice sandstone is more monotonous when compared with very similar but older Istebna beds. The size of pebbles and grains is smaller, sorting poorer, but mechanical abrasion better. In many beds exhibiting grading and lensing, and in regularly ungraded beds, orientation of pebbles may be noticed in the form of imbrication. The degree of orientation is better or worse, depending on the size, form and weight of pebbles and on the hydrodynamics of the current. Angles of longer axes and of planes of pebbles are fairly high (most frequently 20°-35°). The direction of transport by currents, as read from the inclination of pebbles, is toward the northeast (azimuth 60° on the average), a fact corroborated by the direction of flow marks (Fig. 4); but the dispersion of direction is fairly large.

Features pointing to a strong intraformational erosion are, as parallel current bedding, erosional hieroglyphs and intrastratal wash-outs,

very frequent in the Ciężkowice sandstone. Clay balls and shaly fragments as well as lensing occurring in sandy beds also point to erosion.

There are also features which may be referred to some submarine mass movements on the bottom in the still unconsolidated sandy beds (Fig. 1d). It is possible that the irregular, ungraded bedding was formed by some more violent sliding of deposited material (Fig. 1c).
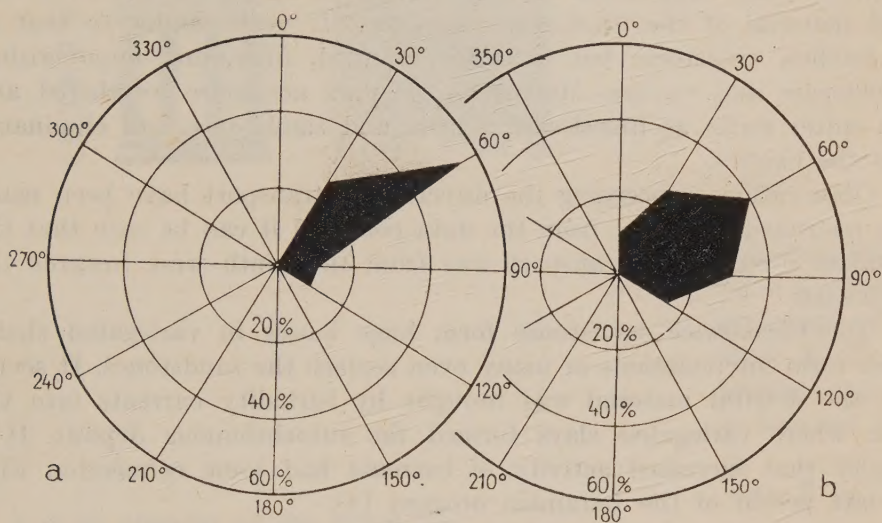

Fig. 4. Preferred orientation of pebbles (a) and flow marks (b)

It follows from the above that the Ciężkowice sandstone is characterised by a very restless type of sedimentation in comparison with other Flysch series. The turbidity currents which deposited the layers of the Ciężkowice sandstone must have been stronger, of rather lower density, but greater velocity than the currents depositing other Flysch beds. This is suggested by many signs of intraformational erosion, very poor separation of grades in the sandstones, and a lack of conglomerates with larger pebbles. The depositing currents probably, in most instances, dragged the material along the bottom, causing a good orientation of particles, but in some cases the material probably dropped from the currents flowing at a height above the bottom, and was deposited without any orientation. Different inclinations of pebbles reflect to some degree the changes in velocity and density of currents. It seems that the current bedding in coarse-grained beds, more frequent in the Ciężkowice sandstone than in any other Flysch beds, was formed by swift currents of low density, while muddy conglomerates could have been deposited by currents approaching mudflows. Strong erosive action of currents could produce erosion of large packets of feebly consolidated sandstones from the bottom or from the area where the currents were generated; owing to this, lenti-

cular bedding was probably formed. Fragments of shales and clay balls were transported in smaller quantities.

Very monotonous composition of sandstones and fairly good abrasion of grains suggests that density currents were carrying material deriving from older sediments. It is possible, that, at least to some extent, they eroded the Istebna sandstones, deposited shortly before the Ciężkowice sandstone. This supposition may be supported by the fact that the detrital material of the Ciężkowice sandstone is very similar to that of the Istebna sandstone, but is better rounded, finer, and impoverished in feldspars and exotics. Moreover, feldspars are more weathered and such softer rocks as limestones, schists and sandstones, are eliminated from the exotics.

Observations concerning the direction of transport have been made on a restricted area, but from the data collected it can be seen that the prevalent direction of transport was from the south-west towards the north-east.

The Ciężkowice sandstones form huge lenses in variegated shales which form intercalations or many even replace the sandstones. It seems that the detrital material was brought by turbidity currents into the basin where variegated clays formed the autochthonous deposit. It is possible that increased activity of currents had some connection with the late period of the Laramian orogeny [4].

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